Rigid meromorphic cocycles for orthogonal groups

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These are some rough notes from a study group at the Max Planck Institute in Bonn, autumn 2024. The topic discussed is rigid meromorphic cocycles for orthogonal groups, closely following the paper [DGL23].

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1 Background: rigid meromorphic cocycles for SL_2

2 Lecture 2: Orthogonal groups and symmetric spaces

Let $V/\mathbb{Q}$ be a vector space with a non-degenerate symmetric bilinear form

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{Q}. \tag{2.1}$$

Then $q(v) := \frac{1}{2}\langle v, v \rangle$ is a quadratic form. We define the corresponding orthogonal groups

$$O_V := \{g \in \mathrm{GL}(V) : q(gv) = q(v)\} \quad \text{and} \quad \mathrm{SO}_V := O_V \cap \mathrm{SL}(V). \tag{2.2}$$

We may diagonalize the form q over $\mathbb{R}$, and we say V has (real) signature (r, s) if q is equivalent to

$$\sum_{j=1}^{r} x_j^2 - \sum_{j=1}^{s} x_{r+j}^2 \tag{2.3}$$

over $\mathbb{R}$. We let $n = r + s$ denote the dimension of V.

2.1 Archimedean symmetric spaces

Definition 2.1: The *archimedean symmetric space* of X_∞ is the set of maximal negative definite subspaces of $V_\mathbb{R} := V \otimes \mathbb{R}$.

One can prove that the dimension of X_∞ is $r \cdot s$.

Lemma 2.2: The group $O_V(\mathbb{R})$ acts transitively on X_∞.

Proof: Let z and z' be elements of X_∞, and view them as subspaces of $V_\mathbb{R}$ with the induced quadratic form. Since quadratic spaces over $\mathbb{R}$ are determined by their signature up to isometry, and both have signature $(0, s)$, there exists an isometry $z \to z'$. By Witt's extension theorem this extends to an isometry $V_\mathbb{R} \to V_\mathbb{R}$. □

Fix a point $z_0 \in X_\infty$. The lemma implies that we may identify X_∞ with $O_V(\mathbb{R}) / \mathrm{Stab}_{O_V(\mathbb{R})} z_0$.

Example 2.3:

- *Suppose V has signature $(r, 0)$. Then X_∞ is simply a point.*
- *Suppose V has signature $(r, 1)$. Over $\mathbb{R}$, q is equivalent to $q_\mathbb{R}(x) := x_1^2 + \ldots + x_r^2 - x_{r+1}^2$. If $q_\mathbb{R}(x) < 0$, then $x_1^2 + \ldots + x_r^2 < x_{r+1}^2$. Since we are interested in the line spanned by x, we may rescale so that $x_{r+1} = 1$. Then the line corresponds to a unique point $(x_1, \ldots, x_r) \in \mathbb{R}^r$ with*

$$x_1^2 + \ldots + x_r^2 < 1. \tag{2.1}$$

This implies that X_∞ can be identified with the unit ball in $\mathbb{R}^r$. Note that the topology is not the subspace topology, but rather the hyperbolic topology.

Example 2.4: *Let V be of signature $(r, 2)$. For any field $K/\mathbb{Q}$, we define the* quadric of isotropic lines over K *to be*

$$\mathcal{Q}(K) := \{v \in V_K - \{0\} : q_K(v) = 0\}/K^\times. \tag{2.2}$$

This is a closed subvariety of $\mathbb{P}^1(V)$. *We now define*

$$\tilde{X}_\infty := \{[v] \in \mathcal{Q}(\mathbb{C}) : \langle v, \overline{v} \rangle < 0\}, \tag{2.3}$$

which is an open subset of $\mathcal{Q}(\mathbb{C})$. *The involution* $x \mapsto \overline{x}$ *exchanges the two connected components of* $\tilde{X}_\infty$. *Given a line* $[v] \in \tilde{X}_\infty$, *write* $v = v_1 + iv_2$. *Then one can check that* $q_{\mathbb{R}}(v_1) = q_{\mathbb{R}}(v_2) = 0$, *so* $\mathbb{R}v_1 + \mathbb{R}v_2 \in X_\infty$. *This gives a* 2*-to-*1 *cover* $\tilde{X}_\infty \to X_\infty$. *In particular, this gives* X_∞ *the structure of a complex manifold. This is specific to the signature* $(r, 2)$ *setting; in general signature there is no complex structure on* X_∞.

We can also define

$$\tilde{X}'_\infty := \{[v] \in \mathcal{Q}(\mathbb{C}) : \langle v, w \rangle \neq 0 \text{ for all } [w] \in \mathcal{Q}(\mathbb{R})\}. \tag{2.4}$$

This natural contains $\tilde{X}_\infty$.

Exercise 2.5: Show that $\tilde{X}'_\infty = \tilde{X}_\infty$ unless $r = 2$.

2.2 p-adic symmetric spaces

In this section we will assume $n \geq 3$, and fix $p \geq 3$. We define $\mathbb{C}_p$ to be the completion of a fixed algebraic closure $\overline{\mathbb{Q}}_p$.

Suppose $V_{\mathbb{Q}_p}$ contains a self-dual $\mathbb{Z}_p$-lattice Λ. Then q induces a non-degenerate $\mathbb{F}_p$-valued pairing on $\Lambda / p\Lambda$. By the Chevalley–Warning theorem, this form has a zero, which lifts to an isotropic vector in Λ by Hensel's lemma. It follows that $\mathcal{Q}(\mathbb{Q}_p)$ is non-empty. Inspired by the definition of $\tilde{X}'_\infty$, we have the following:

Definition 2.6: The *p-adic symmetric space of* O_V is

$$X_p := \left\{[v] \in \mathcal{Q}(\mathbb{C}_p) : \langle v, w \rangle \neq 0 \text{ for all } w \in \mathcal{Q}(\mathbb{Q}_p)\right\}. \tag{2.1}$$

Proposition 2.7: *The space* X_p *carries the structure of a rigid analytic variety.*

Proof: For any line $[w] \in \mathcal{Q}(\mathbb{Q}_p)$, we may find $w' \in \Lambda' := \Lambda - p\Lambda$ such that $[w] = [w']$. Similarly, $[v] \in \mathcal{Q}(\mathbb{C}_p)$, let v' be a corresponding vector in $\Lambda_{\mathbb{C}_p} - \mathfrak{m}_{\mathbb{C}_p}\Lambda_{\mathbb{C}_p}$. We extend the valuation on $\mathbb{Q}_p$ to $\mathbb{C}_p$, and so for $k \in \mathbb{N}$ the set

$$X^{\leq k}_{p,\Lambda} := \left\{v \in \mathcal{Q}(\mathbb{C}_p) : \mathrm{ord}_p\langle v', w' \rangle \leq k \text{ for all } [w] \in \mathcal{Q}(\mathbb{Q}_p)\right\} \tag{2.2}$$

is well-defined. Then $X_p = \bigcup_k X^{\leq k}_{p,\Lambda}$, and one can show that $X^{\leq k}_p$ is an affinoid open. □

Note that while the choice of basic affinoids depends on Λ, the space X_p itself is independent.

Example 2.8: *Suppose V has real signature* $(1, 2)$. *We claim that*

$$X_p \cong \mathfrak{h}_p = \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p). \tag{2.3}$$

3 Lecture 3: Special cycles

In this lecture, the goal is to construct certain divisors on X_∞ and X_p. We start with X_∞.

3.1 Archimedean divisors

Definition 3.1: Fix $v \in V_\mathbb{R}$ with $q(v) > 0$, and define

$$\Delta_{v,\infty} := \{z \in X_\infty : z \subset v^\perp\}, \tag{3.1}$$

where $v^\perp$ is the orthogonal complement of the span of v in V.

This can be identified with the symmetric space of the orthogonal group of $v^\perp \subset V_\mathbb{R}$.

Remark 3.2:

(i) *Define* $\mathcal{T} := \{(w, z) \in V_\mathbb{R} \times X_\infty : w \in z\}$. *This is naturally a vector bundle over X_∞ via projection onto the second factor. Taking* $\mathrm{pr}_z : V \to z$ *to be the orthogonal projection, we obtain a section* $s_v : X_\infty \to \mathcal{T}$ *given by* $z \mapsto (\mathrm{pr}_z(v), z)$. *Then $\Delta_{v,\infty}$ is the preimage of the* 0 *-section* $(0, z)$ *under v.*

(ii) *The action of* $G(\mathbb{R})$ *lifts to* $\mathcal{T}$, *and using this it is easy to verify that* $g \cdot \Delta_{v,\infty} = \Delta_{gv,\infty}$.

3.2 p-adic divisors

We now turn to the divisors on X_p.

Definition 3.3: Let $v \in V_{\mathbb{Q}_p}$ be a vector with $q(v) \neq 0$, i.e. v is anisotropic. By analogy with the archimedean setting, we define

$$\Delta_{v,p} := \{\xi \in X_p : \xi \subset v^\perp\}, \tag{3.1}$$

called a *special divisor* on X_p.

Then for any $g \in G(\mathbb{Q}_p)$ we have $g \cdot \Delta_{v,p} = \Delta_{gv,p}$. Recall that a *hyperbolic plane* is a 2-dimensional quadratic space with quadratic form $q(x, y) = x \cdot y$. A *hyperbolic space* $\mathbb{H}$ is a direct sum of hyperbolic planes.

Example 3.4: *Suppose* $V_{\mathbb{Q}_p} \cong \mathbb{Q}_p v \cdot \mathbb{H}$. *Then* $\Delta_{v,p}$ *is trivial. More generally, if* V *is a quadratic space with*

$$q(x) = x_1^2 + x_2^2 + x_3^2, \tag{3.2}$$

then $\Delta_{v,p}$ *is trivial if and only if* $q(v)$ *is a square in* $\mathbb{Q}_p^\times$.

Recall that we fixed a self-dual lattice Λ in $V_{\mathbb{Q}_p}$. To understand the intersections of $\Delta_{v,p}$ with the basic affinoids $X_{p,\Lambda}^{\leq k}$, we first relate v and Λ.

Definition 3.5: Let $v \in V_{\mathbb{Q}_p}$. Then we define the *order of* v *with respect to* Λ to be

$$\operatorname{ord}_\Lambda(v) := \sup\left\{\ell \in \mathbb{Z} : \frac{v}{p^\ell} \in \Lambda\right\} \in \mathbb{Z} \cup \{\infty\}. \tag{3.3}$$

We also define the *isotropy level*

$$\operatorname{iso}_\Lambda(v) := \operatorname{ord}_p\big(q(v)\big) - 2\operatorname{ord}_\Lambda(v). \tag{3.4}$$

In other words, $\operatorname{iso}_\Lambda(v) = \operatorname{ord}_p\big(q(v_0)\big)$ if $v = p^\ell v_0$ with $v_0 \in \Lambda' := \Lambda - p\Lambda$.

Lemma 3.6: Fix an anisotropic vector $v \in V_{\mathbb{Q}_p}$, and let $k_v = \operatorname{iso}_\Lambda(v)$. Then:
(i) for any $\varepsilon > 0$, the intersection $\Delta_{v,p} \cap X_{p,\Lambda}^{k_v - \varepsilon}$ is empty.
(ii) If $v^\perp$ is not a hyperbolic space, then

$$\Delta_{v,p} \cap X_{p,\Lambda}^{\leq \lceil 3k_v/2 \rceil} \neq \emptyset. \tag{3.5}$$

[TODO: Insert drawing]

Corollary 3.7: *Fix* $m \in \mathbb{Q}_p^\times$ *and* $k > 0$. *If* $v \in V_{\mathbb{Q}_p}$ *with* $q(v) = m$ *such that* $\Delta_{v,p} \cap X^{\leq k}$, *then* $v \in p^{-\ell}\Lambda$ *for* $\ell \leq \frac{1}{2}\big(k - \operatorname{ord}_p(m)\big)$.

Proof: TODO: fill in □

3.3 Locally finite divisors

In this subsection, we combine the two above constructions. Fix a $\mathbb{Z}[1/p]$-lattice L in V. Let Γ be a subgroup of SO_V which stabilises Λ. Such a group is called a p-arithmetic *p-arithmetic subgroup of* SO_V. This is a discrete subgroup of $\mathrm{SO}_V(\mathbb{R}) \times \mathrm{SO}_V\big(\mathbb{Q}_p\big)$.

The construction of mixed divisors relies on the following set of data:

(i) a compact subset $C \subset X_\infty$,
(ii) a finite subset $S \subset \mathbb{Q}_{>0}$,
(iii) if V^+ denotes the set of positive vectors, a set of integers $(a_v) \in \mathbb{Z}^{V^+}$ satisfying:
  - $a_{\gamma v} = a_v$ for all $\gamma \in \Gamma$,
  - $a_v = 0$ if $\Delta_{v,\infty} \cap C = \emptyset$ or $q(v) \notin S$.

Definition 3.8: The formal sum

$$\Delta := \sum_{v \in V^+} a_v \cdot \Delta_{v,p} \tag{3.1}$$

is called a *locally rational finite quadratic divisor* in X_p.

Note that for any basic affinoid $\mathcal{A}$,

$$\Delta \cap \mathcal{A} := \sum_{\substack{v \in V^+ \\ \Delta_{v,p} \cap \mathcal{A} \neq \emptyset}} a_v \Delta_{v,p} \tag{3.2}$$

is a finite formal sum. Indeed, Corollary 3.7, the set

$$\left\{ v \in V^+ : \Delta_{v,\infty} \cap C, q(v) \in S \text{ and } \Delta_{v,p} \cap X_{p,\Lambda}^{\leq k} \right\} \tag{3.3}$$

is both compact and discrete, hence finite.

4 Lecture 4-5: Kudla–Millson divisors

In this lecture, we will first turn back to the case of $\mathrm{SL}_2(\mathbb{Z}[1/p])$ to motivate the ensuing constructions.

4.1 Modular symbols

Let Ω be an abelian group. An Ω*-valued modular symbol* is a function $m : \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \to \Omega$ satisfying:
(i) m is *alternating*, $m(r,s) = -m(s,r)$,
(ii) m is *additive*, $m(r,s) + m(s,t) = m(r,t)$,

for all $r, s, t \in \mathbb{P}^1(\mathbb{Q})$. We denote the set of such functions by $\mathrm{MS}(\Omega)$. We can also describe this in terms of divisors: let $\mathrm{Div}\, \mathbb{P}^1(\mathbb{Q}) = \mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]$ be the group of divisors on $\mathbb{P}^1(\mathbb{Q})$, and let $\mathrm{Div}^0\, \mathbb{P}^1(\mathbb{Q})$ be the degree zero divisors, the kernel of the augmentation map $\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})] \to \mathbb{Z}$. Then $\mathrm{MS}(\Omega)$ is equal to the set $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}^0\, \mathbb{P}^1(\mathbb{Q}), \Omega)$.

If Γ is a group acting on $\mathbb{P}^1(\mathbb{Q})$ and Ω is a Γ-module, then we define the Γ-*invariant modular symbols* to be

$$\begin{aligned} \mathrm{MS}(\Omega)^\Gamma &= \mathrm{Hom}_\Gamma\big(\mathrm{Div}^0\,\mathbb{P}^1(\mathbb{Q}),\Omega\big) \\ &= \big\{m \in \mathrm{MS}(\Omega) : m(\gamma s, \gamma t) = \gamma \cdot m(s,t) \text{ for all } s,t \in \mathbb{P}^1(\mathbb{Q}), \gamma \in \Gamma\big\}. \end{aligned} \tag{4.1}$$

Let $\Gamma_\infty \subset \Gamma$ be the stabiliser of $\infty \in \mathbb{P}^1(\mathbb{Q})$. We define

$$H^1_{\mathrm{par}}(\Gamma,\Omega) = \ker\big(H^1(\Gamma,\Omega) \to H^1(\Gamma_\infty,\Omega)\big), \tag{4.2}$$

where the map is induced from the inclusion $\Gamma_\infty \to \Gamma$. The elements of $H^1_{\mathrm{par}(\Gamma,\Omega)}$ are called *parabolic* cocycle classes. These may frequently be described in terms of modular symbols:

Lemma 4.1: Suppose $\Omega^\Gamma = \Omega^{\Gamma_\infty}$. Then

$$\mathrm{MS}(\Omega)^\Gamma \cong H^1_{\mathrm{par}}(\Gamma,\Omega). \tag{4.3}$$

Proof ((sketch)): Let m be a modular symbol, and define $\varphi(\gamma) := m(\infty, \gamma\infty)$. Then

$$\begin{aligned} \varphi(\gamma\gamma') &= m(\infty, \gamma\gamma'\infty) \\ &= m(\infty,\gamma\infty) + m(\gamma\infty, \gamma\gamma'\infty) \\ &= \varphi(\gamma) + \gamma\varphi(\gamma'\infty). \end{aligned} \tag{4.4}$$

Since m is alternating, φ is parabolic. Note that H^1_{par} has no coboundaries: [insert proof]. □

Corollary 4.2: *Let* $\Gamma = \mathrm{SL}_2(\mathbb{Z}[1/p])$ *and let* $\mathcal{M}^\times$ *be the multiplicative group of rigid meromorphic functions on* $\mathfrak{h}_p$*, with the weight* 0 *action of* Γ*. Then*

$$H^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times) \cong \mathrm{MS}(\Omega)^\Gamma. \tag{4.5}$$

Now fix $r, s \in \mathbb{P}^1(\mathbb{Q})$, and let $\mathrm{geo}(r,s) \subset \mathfrak{h}$ be the unique oriented geodesic in the *complex* upper half plane from r and s. Similarly, if $w \in \mathbb{P}^1(\mathbb{R})$ is a real quadratic point, we denote by $\mathrm{geo}(w) \subset \mathfrak{h}$ the oriented geodesic from w to its conjugate w'. We may then define the *oriented intersection number* $\langle \mathrm{geo}(r,s), \mathrm{geo}(w)\rangle$ between $\mathrm{geo}(r,s)$ and $\mathrm{geo}(w)$ as follows: fix an orientation on the upper half plane, or equivalently, a compatible oriented basis for the tangent plane at each point $z \in \mathfrak{h}$. Since $r, s \in \mathbb{P}^1(\mathbb{Q})$, one can easily show that $\mathrm{geo}(r,s)$ and $\mathrm{geo}(w)$ intersect transversely in at most one point $z \in \mathfrak{h}$. Then the intersection number is 1 if the induced basis on $T_z\,\mathrm{geo}(r,s) \times T_z\,\mathrm{geo}(w)$ has the same orientation as $T_z\mathfrak{h}$, and -1 otherwise.

Lemma 4.3: Let $\tau \in \mathbb{P}^1(\mathbb{R})$ be a real quadratic point. Then

$$\Delta_\tau(r,s) := \sum_{w \in \mathrm{SL}_2(\mathbb{Z})\cdot\tau} \langle \mathrm{geo}(r,s), \mathrm{geo}(w)\rangle \cdot [w] \tag{4.6}$$

is an $\mathrm{SL}_2(\mathbb{Z})$-invariant modular symbol valued in $\mathbb{P}^1(\mathbb{R})$.

Proof: Note that additivity and alternation is clear from the definition of the intersection number. However, we ought to check that the sum is in fact finite. Since $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$, it suffices to check this for $r = 0$ and $s = \infty$. Then the statemnet is reduced to showing that there are finitely many w for which $w > 0 > w'$, which follows from an elementary argument. □

Now we will replace $\mathrm{SL}_2(\mathbb{Z})$ with $\mathrm{SL}_2(\mathbb{Z}[1/p])$. The same proof, combined with Corollary 4.2, shows that

$$\varphi(\gamma) := \sum_{w \in \mathrm{SL}_2(\mathbb{Z})\cdot\tau} \langle \mathrm{geo}(\infty, \gamma\infty), \mathrm{geo}(\tau)\rangle \cdot \frac{1}{z-w} \tag{4.7}$$

is a parabolic cocycle in $H^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times)$. To construct rigid meromorphic cocycles, Darmon and Vonk take the preimage under $\mathrm{d}\log : H^1_f(\Gamma, \mathcal{M}^\times) \to H^1_{\mathrm{par}}(\Gamma, \mathcal{M})$, giving an infinite product of the form

$$\varphi(\gamma) = \prod_{w \in \Gamma\tau} (z-w)^{\langle \mathrm{geo}(\infty,\gamma\infty), \mathrm{geo}(w)\rangle} \tag{4.8}$$

4.2 From signature $(2,1)$ to $(r,1)$

We now put the results of the previous section into the context of orthogonal groups.

Example 4.4: *Take* $V = \mathrm{Mat}_2(\mathbb{Q})^{\mathrm{Tr}=0}$ *and* $g(v) = -\det(v)$ *of signature* $(2,1)$*, and recall that* $G := \mathrm{SO}_V$ *is natural identified with* PGL_2*, acting by* $g \cdot v = gvg^{-1}$*. The bilinear form is* $\langle v, w\rangle = \mathrm{Tr}(vw^\#)$*, where*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\# := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{4.1}$$

is the adjugate.

There are two ways of identifying the corresponding symmetric space X_∞ *with* $\mathfrak{h}$*. The first is to note that the map* $z \mapsto z^\perp$ *identifies* X_∞ *with the symmetric space* X'_∞ *for* $\mathrm{SO}(1,2)$ *whose under-*

lying quadratic space is $(V,-q)$*. This may be described directly as in* Example 2.4*. Namely, we have*

$$\tilde{X}_\infty = \left\{[w] \in \mathcal{Q}(\mathbb{C}) : -\operatorname{Tr}\left(w\overline{w}^\#\right) < 0\right\}, \tag{4.2}$$

where as before $\mathcal{Q}$ *is the quadric of isotropic lines in* $\mathbb{P}(V)$*. Note that* $w \in V_\mathbb{C} - \{0\}$ *is isotropic if and only if it has rank* 1 *as a matrix. Any rank* 1 *matrix is of the form*

$$w = u_1 \cdot u_2^t \tag{4.3}$$

for some vectors u_1, u_2 *in* $\mathbb{R}^2$*.*[1] *Rescaling, and using* $\operatorname{Tr}(w) = 0$*, we may take*

$$w = \begin{pmatrix} \tau \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\tau \end{pmatrix} = \begin{pmatrix} \tau & -\tau^2 \\ 1 & -\tau \end{pmatrix}. \tag{4.4}$$

Furthermore, $[w] \in \tilde{X}_\infty$ *exactly when* $(\tau - \overline{\tau})^2 > 0$*, i.e.* $\Im(\tau) \neq 0$*, and the connected component* X_∞ *corresponds to* $\mathfrak{h}$*. It is easy to check that the map* $w \mapsto \tau$ *is* $\mathrm{PGL}_2(\mathbb{R})$*-equivariant. Under this map, note that* $\mathcal{Q}(\mathbb{Q})$ *maps to* $\mathbb{P}^1(\mathbb{Q})$*.*

This suggests that $\mathcal{Q}(\mathbb{Q})$ might be the right analogue for $\mathbb{P}^1(\mathbb{Q})$ in the case of $O(r,1)$.

Definition 4.5: Let Ω be a right G-module. Then we define

$$\mathrm{MS}(\Omega) := \operatorname{Hom}_\mathbb{Z}(\mathbb{Z}[\mathcal{Q}(\mathbb{Q})]_0, \Omega). \tag{4.5}$$

Fix a p-arithmetic subgroup $\Gamma \subset \mathrm{SO}_V$ as before. Then by adapting the proof of Corollary 4.2 one can show:

Lemma 4.6: There is an injective map

$$\mathrm{MS}\left(\mathrm{Div}^\dagger_{\mathrm{rq}} X_p\right)^\Gamma \to H^1\left(\Gamma, \mathrm{Div}^\dagger_{\mathrm{rq}} X_p\right). \tag{4.6}$$

Next, our goal is to try to construct elements of the left-hand side using analogues of modular symbols.

Definition 4.7: Fix a pair of lines $\ell_-, \ell_+ \in \mathcal{Q}(\mathbb{Q})$, and let

$$\left[\ell_-, \ell_+\right] := \left\{z \in X_\infty : z \subset \ell_- \oplus \ell_+\right\}. \tag{4.7}$$

[1] Indeed, if w has rank 1, fix a vector u in the image of w, unique up to scaling. Then for any $v \in \mathbb{C}^2$, $w(v) = \lambda(v) \cdot u$ for some linear functional $\lambda(v)$. Writing $\lambda(v) = \langle v, u_2\rangle$, we find $Av = \left(u_1 \cdot u_2^t\right)v$.

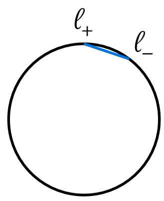
This is called *the geodesic from* ℓ_- *to* ℓ_+.

Note that it is naturally a one-dimensional submanifold of X_∞.

Example 4.8: *Suppose* $V = \mathbb{R}^3$ *with quadratic form* $x^2 + y^2 - z^2$, *and identify* X_∞ *with the open unit disk* $\{|x| < 1\} \subset \mathbb{R}^2$ *via the map* $[x : y : z] \mapsto (x/z, y/z)$ *for* $z \neq 0$.

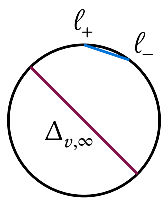
Define the lines $\ell_- = [3 : 4 : 5]$ *and* $\ell_+ = [0 : 1 : 1]$ *in* $\ell_\pm \in \mathcal{Q}(\mathbb{Q})$. *They correspond to the points* $\left(\frac{3}{5}, \frac{4}{5}\right)$ *and* $(0, 1)$ *on the unit circle. Then we have a parametrisation*

$$[\ell_-, \ell_+] = \left\{\left(\frac{3a}{5a+1}, \frac{4a+1}{5a+1}\right) : a \in \mathbb{R}\right\}. \tag{4.8}$$



Fix a vector $v \in V$ of positive norm. Recall that $\Delta_{v,\infty} = \{z \in X_\infty : z \subset v^\perp\}$ is a subspace of X_∞. If $v^\perp$ is anisotropic, then $\Delta_{v,\infty}$ does not interesect $\mathcal{Q}(\mathbb{Q})$, and so the intersection $[\ell_-, \ell_+] \cap \Delta_{v,\infty}$ is transversal, and we may define the oriented intersection number as earlier.

Example 4.9: *Continuing the previous example, let* $v = (1, 1, 0)$, *and a quick computation shows that* $v^\perp = \{x + y = 0\}$, *which is anisotropic. Then* $\Delta_{v,\infty} = \{(x, -x) : x^2 < \frac{1}{2}\}$,



and so $[\ell_-, \ell_+] \cap \Delta_{v,\infty} = 0$.

Remark 4.10: *The condition that v be anisotropic is usually not satisfied. For example, if* $r \geq 5$, *then* $v^\perp$ *has signature* $(4, 1)$, *and so the quadratic form represents zero by Lagrange's four square theorem.*

Lemma 4.11: Let $d > 0$, and let $L \subset V$ be a $\mathbb{Z}$-lattice. Then for any fixed $\ell_\pm \in \mathcal{Q}(\mathbb{Q})$, the set

$$\left\{v \in L : q(v) = d \text{ and } \Delta_{v,\infty} \cap [\ell_-, \ell_+] \neq \emptyset\right\} \tag{4.9}$$

is finite.

Proposition 4.12: *Let* $\Gamma \subset O_V$ *be a p-arithmetic subgroup, and let* $\mathcal{O}$ *be a finite union of* Γ*-orbits such that for any* $v \in \mathcal{O}$, $v^\perp$ *is anisotropic. Then for any* $\ell_\pm \in \mathcal{Q}(\mathbb{Q})$, *the divisor*

$$\sum_{v \in \mathcal{O}} (\Delta_{v,\infty} \cap [\ell_-, \ell_+])\Delta_{v,p} \in \operatorname{Div}_{\mathrm{rq}}^\dagger X_p \tag{4.10}$$

is a locally finite Γ*-invariant rational quadratic divisor. Consequently, the map*

$$\tilde{\mathscr{D}}_{\mathcal{O}} : \mathcal{Q}(\mathbb{Q}) \times \mathcal{Q}(\mathbb{Q}) \to \operatorname{Div}_{\mathrm{rq}}^\dagger X_p \tag{4.11}$$

defines a Γ*-invariant modular symbol.*

Applying Lemma 4.6 then gives a cocycle in $H^1\left(\Gamma, \operatorname{Div}_{\mathrm{rq}}^\dagger X_p\right)$.

4.3 From signature $(r, 1)$ to (r, s)

For $s = 0$, our quadratic space V is positive definite, and consequently X_∞ is compact. On the other hand, a p-arithmetic subgroup Γ of SO_V acts discretely on X_p. It is then natural to study Γ-invariant divisors on $\Gamma \setminus X_p$, which we may interpret as classes in $H^0\left(\Gamma, \operatorname{Div}_{\mathrm{rq}}^\dagger X_p\right)$.

Based on this, it is natural to guess that in real signature (r, s) there ought to be classes in $H^s\left(\Gamma, \operatorname{Div}_{\mathrm{rs}}^\dagger X_p\right)$. To construct these, we will consider maps $C_s(X_\infty) \to \operatorname{Div}_{\mathrm{rq}}^\dagger X_p$ of the form

$$c \mapsto \sum_{v \in \mathcal{O}} (c \cap \Delta_{v,\infty}) \cdot \Delta_{v,p}. \tag{4.1}$$

However, in general these intersections may not be transverse. There are two ways to remedy this: the first is to define the intersection numbers in terms of the cup product[2]. We take the second approach, which is more ad-hoc: fix $v \in V^+$, and consider the diagram

$$\begin{array}{ccccc} C_s(X_\infty) & \xrightarrow{\;d\;} & \ker(d_{s-1} : C_{s-1}(X_\infty) \to C_{s-2}(X_\infty)) & \longrightarrow & H_{s-1}(X_\infty) \to 0 \\ \uparrow & \searrow & \uparrow & & \uparrow \\ C_s(X_\infty - \Delta_{v,\infty}) & \longrightarrow & \ker\left(d_{s-1} : C_{s-1}\left(X_\infty - \Delta_{v,\infty}\right) \to C_{s-2}\left(X_\infty - \Delta_{v,\infty}\right)\right) & \longrightarrow & H_{s-1}\left(X_\infty - \Delta_{v,\infty}\right) \to 0 \end{array}$$

Since $X_\infty - \Delta_{v,\infty}$ is homotopic to S^{s-1}, the bottom right entry is isomorphic to $\mathbb{Z}$. Let $\mathfrak{C}_s(X_\infty)$ be the subcomplex of $C_s(X_\infty)$ consisting of chains c such that

[2] This requires some effort, as X_∞ is contractible (hence cup products are 0) and $\Gamma \setminus X$ is not nice.

$$dc \in \ker\big(d_{s-1} : C_{s-1}(X_\infty - \Delta_{v,\infty}) \to C_{s-2}(X_\infty - \Delta_{v,\infty})\big), \tag{4.2}$$

for all $v \in V^+$. Informally, we can think of these as all the chains which intersect $\Delta_{v,\infty}$ transversely. Given $\mathcal{O} \subset V$ a Γ-orbit, we formally obtain a Γ-cocycle

$$\mathcal{D}_{\mathcal{O}} \in H^s\big(\Gamma, \mathrm{Div}^\dagger_{\mathrm{rq}} X_p\big). \tag{4.3}$$

They claim (see §2.4.2) that this is "given by"

$$c \mapsto \sum_{v \in \mathcal{O}} (c \cap \Delta_{v,\infty})\Delta_{v,p}, \tag{4.4}$$

when $c \in \mathfrak{C}_s(X_\infty)$, but this is not entirely precise. The key observation is that $\mathfrak{C}_s(X_\infty)$ is a resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\Gamma]$-module, so in the derived category of Γ-modules we get a map $\mathbb{Z} \to \mathrm{Div}^\dagger[-s]$, which is the same as a Γ-cocycle.

Write $\mathbb{A}^{p,\infty}$ for the adeles away from p and ∞, and let $V^{p,\infty} := V \otimes \mathbb{A}^{p,\infty}$. Let $\mathcal{S}(V^{p,\infty})$ denote the tensor product of local $\mathbb{Z}$-valued Schwartz–Bruhat functions. This has a natural action of Γ by precomposition, and for fixed $\varphi \in \mathcal{S}(V^{p,\infty})$, $m \in \mathbb{Q}$ and $r \in \mathbb{Z}$ we define the finite union of Γ-orbits,

$$\mathcal{O}(m, \varphi, r) = \{v \in V : q(v) = m, \varphi(v) = r\}. \tag{4.5}$$

Note that this is empty for almost all r, since φ can only take finitely many values. This makes the following sum well-defined:

Definition 4.13: A "Kudla–Millson"-divisor is the rational quadratic divisor on X_p given by the expression

$$\mathcal{D}_{m,\varphi} = \sum_{r \in \mathbb{Z}} r \cdot \mathcal{D}_{\mathcal{O}(m,\varphi,r)}. \tag{4.6}$$

In Borcherds-like applications, it is useful to fix a $\mathbb{Z}[1/p]$-lattice L such that Γ acts trivially on the discriminant module $D_L := L^\vee / L$. For $\beta \in D_L$ and $\hat{L} \subset V^{p,\infty}$ the completion of L, note that that $\beta + \hat{L}$ is Γ-invariant, so $\mathbb{1}_{\beta+\hat{L}} \in \mathcal{S}(V^{p,\infty})^\Gamma$. We therefore set

$$\mathcal{D}_{m,\beta} := \mathcal{D}_{\mathcal{O}(m, \mathbb{1}_{\beta+\hat{L}})}. \tag{4.7}$$

5 Appendix: Non-split orthogonal groups

We first consider the situation over $\mathbb{R}$. Let (V, q) be a quadratic space over $\mathbb{R}$ of signature (r, s), with $n = r + s$. After diagonalizing, we can assume q has the shape

$$q(x) = \sum_{j=1}^{r} x_j^2 - \sum_{j=1}^{s} x_{r+j}^2. \tag{5.8}$$

The orthogonal group of V is then naturally identified with

$$O(m,n) := \left\{ g \in \mathrm{GL}_n(\mathbb{R}) : g \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix} g^t = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix} \right\}, \tag{5.9}$$

where I_r is the $r \times r$ identity matrix. Similarly, $\mathrm{SO}(m,n) := O(m,n) \cap \mathrm{SL}_n(\mathbb{R})$.

Proposition 5.1: *Suppose $r, s > 0$. Then the group $\mathrm{SO}(r,s)$ is not connected.*

Proof: The map

$$\mathrm{SO}(r,s) \to \mathbb{R}^\times \text{ given by } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \det(A), \tag{5.10}$$

where the blocks are as in the previous definition, is surjective and continuous, so $\mathrm{SO}(r,s)$ has at least two connected components. □

It turns out that $\mathrm{SO}(r,s)$ has two connected components, and we denote the component of the identity by $\mathrm{SO}^+(r,s)$. We then have the following table of groups and their maximal compact subgroups:[3]

G	$O(r,s)$	$\mathrm{SO}(r,s)$	$\mathrm{SO}^+(r,s)$
K	$O(r) \times O(S)$	$S(O(r) \times O(s))$	$\mathrm{SO}(r) \times \mathrm{SO}(s)$

We can decompose V as $V = V^+ \oplus V^-$, where V^+ is the span of $x_1, ..., x_r$, and V^- is the span of the remaining basis vectors. Then $\mathrm{SO}^+(r,s)$ is precisely the subgroup of $\mathrm{SO}(r,s)$ which preserves the individual orientations on V^+ and V^-.

Example 5.2: *In this example, we consider* $\mathrm{SO}(2,2)$*, which can be described explicitly through its exceptional isogeny with* $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$*. The first hint that such an isogeny might exist comes from the Dynkin diagrams:* $\mathfrak{so}_4$ *has Dynkin diagram* D_2*, which is simply two dots (and no lines), which matches up with the union of the Dynkin diagram for each* $\mathfrak{sl}_2$*, which is a single dot. I think this can be made precise using Satake–Tits diagrams.*

More concretely, fix the vector space $V = \mathrm{Mat}_2(\mathbb{R})$*. One can check that* det *defines a quadratic form on* V*. Furthermore, since*

[3] A reference for this is https://www.math.toronto.edu/mein/teaching/LieClifford/cl12.pdf.

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = \frac{1}{4}\left((a+d)^2 - (a-d)^2 + (b-c)^2 + (b+c)^2\right), \tag{5.11}$$

the space V has signature $(2,2)$. *An orthogonal basis for V is given by*

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad X_+ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.12}$$

Then I and X_+ span V^+, and h and X_- span V^-.

Note that $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$ *acts naturally on V via* $(g_1, g_2) \cdot v = g_1 v g_2^{-1}$. *This gives an embedding* $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R}) \to \mathrm{GL}_4(\mathbb{R})$, *but I don't think it's the natural one. Let's find the elements which preserve* det: *if* $\det(g_1 v g_2^{-1}) = \det(v)$ *for all $v \in V$, then evidently* $\det(g_1) = \det(g_2)$, *and vice versa. Therefore, we obtain a map*

$$\tilde{G} := \mathrm{GL}_2(\mathbb{R}) \times_{\det} \mathrm{GL}_2(\mathbb{R}) \to O_V \cong O(2,2). \tag{5.13}$$

Note that the diagonal matrices act trivially, so the kernel of this map is $\mathbb{R}^\times$. The map is not surjective; consider the map sending I to $-I$ and preserving the other basis vectors. This is certainly an orthogonal transformation, but it is straightforward to check by bashing matrix multiplication this is not encoded by an element of $\tilde{G}$.

We claim that the image of $\tilde{G}$ is in fact $\mathrm{SO}(2,2)$. *One way to see this is by noting that both groups have two connected components, and then showing that the induced maps on Lie algebras is surjective. In other words, we have a short exact sequence*

$$1 \to \mathbb{R}^\times \xrightarrow{\Delta} \mathrm{GL}_2(\mathbb{R}) \times_{\det} \mathrm{GL}_2(\mathbb{R}) \to \mathrm{SO}(2,2) \to 1. \tag{5.14}$$

In fact, this proves that $\tilde{G} = \mathrm{GSpin}_V$. It also gives an explicit description of $\mathrm{SO}^+(2,2)$*; it is isomorphic to*

$$\mathrm{GL}_2(\mathbb{R})^+ \times_{\det} \mathrm{GL}_2(\mathbb{R}) / \mathbb{R}^\times. \tag{5.15}$$

Example 5.3: *We now turn to* $\mathrm{SO}(2,1)$. *Fix the quadratic space* $V = \mathrm{Mat}_2(\mathbb{Q})^{\mathrm{Tr}=0}$ *with quadratic form* $q(v) = \det(v)$. *Since*

$$\det\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a^2 - bc = a^2 + \frac{1}{4}\left((b-c)^2 - (b+c)^2\right), \tag{5.16}$$

this has signature $(2,1)$, *and an orthogonal basis is given by*

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_\pm = \begin{pmatrix} 0 & 1 \\ \mp 1 & 0 \end{pmatrix}, \tag{5.17}$$

so that $\det(X_\pm) = \pm 1$. *If we write*

$$g = a_1 I + a_2 X_+ + a_3 X_-, \tag{5.18}$$

then $\det(g) = a_1^2 + a_2^2 - a_3^2$.

Note that PGL_2 *acts naturally on* V *by matrix conjugation,* $g \cdot v = gvg^{-1}$, *giving a map* $\mathrm{PGL}_2 \to \mathrm{SO}_V$.

Example 5.4: *In this example we find a convenient model for* $\mathrm{SO}(3,1)$. *The punchline is that it is naturally isomorphic to* $\mathrm{PSL}_2(\mathbb{C})$, *viewed as a real Lie group. Intuitively, we can think* $\mathrm{SL}_2(\mathbb{C})$ *as a form of* $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, *so this is compatible with the previous example. NB: I will take a slightly different model from the one in the paper! This seems simpler to me, but there might be a good reason why they chose the other one.*

For $X \in \mathrm{Mat}_2(\mathbb{C})$, *let* $X^\dagger := \overline{X}^t$, *the conjugate transpose. We set*

$$V = \{X \in \mathrm{Mat}_2(\mathbb{C}) : X^\dagger = X\}. \tag{5.19}$$

Explicitly, it consists of matrices

$$X = \begin{pmatrix} a & x + iy \\ x - iy & d \end{pmatrix}, \tag{5.20}$$

so

$$\det(X) = ad - x^2 - y^2 = \frac{1}{4}\left((a+d)^2 - (a-d)^2\right) - x^2 - y^2. \tag{5.21}$$

From this we see that a convenient basis consists of I, h *and* X_-, *as in the previous example, and* $i \cdot X_+$. *Moreover, the form* $q(X) = -\det(X)$ *has real signature* $(3,1)$.

We act by $\mathrm{SL}_2(\mathbb{C})$ *in a similar way as before:* $g \cdot X := gXg^\dagger$. *Note that* $(gXg^\dagger)^\dagger = gX^\dagger g^\dagger = gXg^\dagger$, *so the action on* V *is well-defined.*

Bibliography

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