Étale cohomology seminar: étale sheaves

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1.Sites and Grothendieck topologies

We want to abstract the definition of topology to a relative point of view, e.g the poset of open sets on a topological space X is equivalent to the set of open immersions $U \to X$.

Definition

A *site* $(\mathbf{T}, \mathbf{C})$ on a category $\mathbf{C}$ consists of a set of distinguished maps $(U_i \to U)_{i \in I}$ for each $U \in \mathbf{C}$, called *coverings* of U, satisfying the following axioms

1. If $(U_i \to U)_{i \in I}$ is a covering and $V \to U$ is a map in $\mathbf{C}$, the fiber products $U_i \times_U V$ exist and $(U_i \times_U V)_{i \in I}$ is a covering of V.
2. If $(U_i \to U)_{i \in I}$ is a covering of U, and $(U_{ij} \to U_i)_{j \in J}$ is a covering for each U_i, then $(U_{ij} \to U)$ is a covering of U.
3. $(f : U \to U)$ is a covering of U whenever f is an isomorphism.

The set of coverings on $\mathbf{C}$ is called a Grothendieck (pre-)topology.

Example

If X is a topological space, and let $\mathbf{C}$ be the poset of open subsets. For $U, U' \subseteq V$, $U \times_V U' = U \cap U'$, so we can define $(\phi_i : U_i \to U)$ to be a covering if $\cup \phi_i(U_i) = U$ (the family of maps is *surjective*).

Example

In particular, if X is a scheme, the Zariski site X_{Zar} is defined as above using X as a topological space with the Zariski topology, so the coverings are surjective families of open immersions.

3.Sites and Grothendieck topologies

Example (Small étale site)

The *small étale site* $X_{\text{ét}}$ has as underlying category $\text{Ét}/X$, whose morphisms are étale maps $U \to X$, and whose arrows are morphisms $\phi : U \to V$ in Sch/X:

$$\begin{array}{ccc} U & \xrightarrow{\phi} & V \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

The coverings $(\phi_i : U_i \to U)$ are surjective families $(\cup\phi_i(U_i) = U)$ of étale morphisms in $\text{Ét}/X$. This is well-defined since étale maps are a stable class.

Definition

A *presheaf* of sets on a site $(\mathbf{T}, \mathbf{C})$ is a functor $F : \mathbf{C}^{\text{op}} \to \text{Set}$, i.e. a presheaf on the underlying category. Similarly a presheaf of abelian groups if a functor $F : \mathbf{C}^{\text{op}} \to \text{Ab}$. For a map $f : U' \to U$ in $\mathbf{C}$, and $s \in F(U)$ we denote the image of s under $F(f) : F(U) \to F(U')$ as $s|U'$.

Note that the definition does not depend of the family of coverings.

Definition

A morphism $\phi : F \to F'$ of presheaves if just a natural transformation of functors: maps $\phi(U) : F(U) \to F(U')$ commuting with the restriction maps.

The category $\text{Pshv}(\mathbf{T})$ of presheaves of abelian groups on $\mathbf{T}$ is in fact an abelian category, since (co)kernels can be computed objectwise, and $F \to F' \to F''$ is exact iff it is exact on every U.

Definition

A *sheaf* on a site $\mathbf{T}$ is a presheaf F such that

$$F(U) \xrightarrow{i} \prod_{i \in I} F(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} F(U_i \times_U U_j) \quad (S)$$

is exact for any covering $(U_i \to U)_{i \in I}$. That is, any $s \in F(U)$ is uniquely determined by some $f_i \in F(U_i)$ such that $f_i | U_i \times_U U_j = f_j | U_i \times_U U_j$ for all i, j.

- A morphism of sheaves is defined to be a morphism of presheaves: $\mathrm{Shv}(\mathbf{T})$ is a full subcategory of $\mathrm{PShv}(\mathbf{T})$.
- An étale sheaf is a sheaf on the étale site $X_{\text{ét}}$.
- For an étale sheaf F, $F(\coprod U_i) = \prod F(U_i)$, so setting I empty means that $F(\emptyset) = 0$ (on abelian groups).

6. A criterion to be a sheaf

Proposition

Let F be a presheaf on $X_{ét}$. Then F is a sheaf if it is a sheaf for coverings consisting of open immersions, and if for a covering $(V \to U)$ with both affine $F(U) \to F(V) \rightrightarrows F(V \times_U V)$ is exact.

Proof

- The first condition implies that $F(\coprod_i U_i) = \prod_i F(U_i)$.
- The sequence (S) for a covering $(U_i \to U)_{i \in I}$ with I finite is isomorphic to the sequence for the single map $(\coprod_i U_i \to U)$, since

$$(\coprod_i U_i) \times_U (\coprod_i U_i) = \coprod_{(i,j)} U_i \times_U U_j$$

- Since a finite disjoint union of affines is affine is affine the second condition implies that (S) is exact for $(U_i \to U)$ with I finite and U_i affine.

- Given a covering $(U_j \to U)$, let $U' = \coprod_j U_j$. We want to prove that (S) is exact for $(f : U' \to U)$. Cover $U = \cup U_i$ with open affines, and let $f^{-1}(U_i) = \cup_k U'_{ik}$. Since f is open each $f(U'_{ik})$ is open in U_i, so there is a finite cover $(U'_{ik} \to U_i)_{i \in K_i}$.
- By repeating the process we can assume $U' = \cup U'_{ik}$, $U = \cup U_i$ such that $(U'_{ik} \to U_i)$ is always a finite covering.

$$\begin{array}{ccccc}
F(U) & \longrightarrow & F(U') & \rightrightarrows & F(U' \times_U U') \\
\downarrow & & \downarrow & & \downarrow \\
\prod_i F(U_i) & \longrightarrow & \prod_i \prod_k F(U'_{ik}) & \rightrightarrows & \prod_i \prod_{k,l} F(U'_{ik} \times_U U'_{il}) \\
\downarrow\downarrow & & \downarrow\downarrow & & \\
\prod_{i,j} F(U_i \cap U_j) & \longrightarrow & \prod_{i,j} \prod_{k,l} F(U'_{ik} \cap U'_{jl}) & &
\end{array}$$

Example (Structure sheaf)

For $U \to X$ an étale map define $\mathcal{O}_{X_{\text{ét}}}(U) = \Gamma(U, \mathcal{O}_U)$. This is a sheaf on X_{Zar}, so by the proposition above we only need to check that for a faithfully flat map of rings $A \to B$ (recall $(\text{Spec}(B) \to \text{Spec}(A))$ is flat and surjective hence faithfully flat)

$$0 \to A \to B \xrightarrow{b \mapsto 1 \otimes b - b \otimes 1} B \otimes_A B$$

is exact. (Exercise).

Example (Representable sheaf)

For a X-scheme Z, the presheaf $F(U) := \mathrm{Hom}_X(U, Z)$ is in fact a sheaf. It easy to check it is a sheaf on X_{Zar}. Thus it suffices to check that

$$Z(A) \to Z(B) \rightrightarrows Z(B \otimes_A B)$$

is exact. For $Z = \mathrm{Spec}(R)$ the sequence becomes

$$\mathrm{Hom}_{A-\mathrm{alg}}(R, A) \to \mathrm{Hom}(R, B)_{A-\mathrm{alg}} \rightrightarrows \mathrm{Hom}_{A-\mathrm{alg}}(R, B \otimes_A B),$$

which follows by the exactness of the sequence in the previous slide. Using a patching argument we can extend this to an arbitrary Z. E.g. for $Z = \mathrm{Spec}(\mathbb{Z}[t, t^{-1}]/(t^n - 1)) \times X$ we obtain

$$\mu_n(U) := \mathrm{Hom}_X(U, Z) = \mathrm{Ker}(\Gamma(U, \mathcal{O}_U) \xrightarrow{s \to s^n} \Gamma(U, \mathcal{O}_U)).$$

Example (Constant sheaf)

Let X be a quasi-compact scheme and $A \in \mathrm{Ab}/\mathrm{Set}$. Define the sheaf $\underline{A}$ by $\underline{A}(U \to X) = A^{\pi_0(U)}$ (so functions $U \to A$ constant on each connected component). A map $f : V \to U$ satisfies $f(\pi_0(V)) \subseteq \pi_0(U)$, so the restriction map is defined by precomposition by f.

Example (Locally constant sheaf)

An étale sheaf F is locally constant if for some covering $(U_i \to X)$ $F|U_i$ is constant for each i. Example of locally constant but not constant (for later): for $X = \mathrm{Spec}(k)$, let $M = k'/k$ be a finite separable extension viewed as a G-module.

11. Étale sheaves on Spec(k)

- Let k be a field and $X = \mathrm{Spec}(k)$. Then a presheaf of abelian groups F on $X_{\acute{e}t}$ can be identified with a covariant functor from the category of étale algebras over k: they are of the form $\prod_i^n K_i$ where K_i/k is a finite separable extension of fields.
- Using the criterion to be a sheaf, F is a sheaf iff $F(\prod A_i) = \oplus F(A_i)$ for étale algebras A_i (sheaf on Zariski site, any $U \to \mathrm{Spec}(k)$ is discrete), and if for every finite extension L'/L of finite separable extensions of k

$$F(L) \longrightarrow F(L') \underset{\phi_1}{\overset{\phi_2}{\rightrightarrows}} F(L' \otimes_L L') \quad (*)$$

is exact. This is because in the proof of our criterion, we can restrict to coverings $(V \to U)$ of arbitrarily small affine opens.

Proposition

It is enough to check exactness of $(*)$ *on Galois extensions* L'/L, *and in that case exactness is equivalent to* $F(L) \cong F(L')^G$, *where* $G = \mathrm{Gal}(L'/L)$ *acts on* $F(L')$ *via* $F(\sigma)$ *for* $\sigma : L' \to L'$.

Proof

Suppose first that L'/L is Galois. For $\sigma \in G$ consider

$$L' \underset{x \mapsto x \otimes 1}{\overset{x \mapsto 1 \otimes x}{\rightrightarrows}} L' \otimes_L L' \xrightarrow{\psi_\sigma : x \otimes y \mapsto x\sigma(y)} L',$$

if $z \in F(L')$ is in the equalizer of ϕ_1, ϕ_2 $(*)$ then $F(\sigma)(z) = z$ for all $\sigma \in G$.

Conversely, suppose $z \in F(L')^G$,

$$(\psi_\sigma)_\sigma : L' \otimes_L L' \to \prod_{\sigma \in G} L'$$
$$x \otimes y \mapsto (x\sigma(y))_\sigma$$

is an isomorphism, so $(F(\psi_\sigma))_\sigma$ is injective and thus $\phi_1(z) = \phi_2(z)$. Now suppose that $(*)$ is exact for Galois extensions L''/L. For an arbitrary L'/L let L'' be the Galois closure of L' in L: $L''/L'/L$

$$\begin{array}{ccccc}
F(L) & \longrightarrow & F(L') & \rightrightarrows & F(L' \otimes_L L') \\
\big\downarrow \mathrm{id} & & \big\downarrow & & \big\downarrow \\
F(L) & \longrightarrow & F(L'') & \rightrightarrows & F(L'' \otimes_L L'')
\end{array}$$

The bottom row is exact by assumption. Since $F(L) \to F(L'')$ is injective so is $F(L) \to F(L')$ (and $F(L') \to F(L'')$). After an easy diagram chase the top row is also exact.

14.Étale sheaves on Spec(k)

We have proved: an étale sheaf on Spec(k) can be identified with a covariant functor F from étale algebras to abelian groups satisfying

1. $F(\prod_i A_i) = \oplus_i F(A_i)$ for finitely many étale algebras A_i.
2. $F(L) = F(L')^{\mathrm{Gal}(L'/L)}$ for a finite Galois extension L'/L, where $L', L/k$ are finite separable extensions.

15. Étale sheaves on Spec(k)

Definition

For a profinite group G, a G-module A is *discrete* if the stabilizer of each element is open, i.e. $A = \cup A^U$ for U open subgroups.

Fix a separable closure k^{sep} of k and let $G = \text{Gal}(k^{\text{sep}}/k)$. For a sheaf F define

$$M_F = \varinjlim F(k'),$$

where k' runs over finite separable extensions k'/k. Then G acts on $F(k')$ whenever k'/k is Galois, so it acts on the direct limit. M_F is in fact a discrete G-module, since

$$(M_F)^{\text{Gal}(k^{\text{sep}}/k')} = F(k').$$

Conversely, let M be a discrete G-module. Define a presheaf F_M by

$$F_M(A) = \mathrm{Hom}_G(F(A), M),$$

where $F(A) = \mathrm{Hom}_{k-Alg}(A, k^{\mathrm{sep}})$. For a finite separable extension k'/k

$$G/\mathrm{Gal}(k^{\mathrm{sep}}/k') \cong F(k')$$

as a G-module, so $F_M(k') \cong M^{\mathrm{Gal}(k^{\mathrm{sep}}/k')}$. Under this isomorphism, for $f : k' \to k''$,

$$M^{\mathrm{Gal}(k^{\mathrm{sep}}/k')} \to M^{\mathrm{Gal}(k^{\mathrm{sep}}/k'')}$$

$$x \mapsto \sigma x$$

where $\sigma|k' = f$. F_M is indeed a sheaf:

- $F_M(\prod_i k_i) = \oplus_i F_M(k_i)$ for finite I.
- For k''/k' finite Galois $F_M(k'')^{\mathrm{Gal}(k''/k')} = F_M(k')$.

Proposition

The maps $F \to M_F$ and $M \to F_M$ form an equivalence of categories between the categories of étale sheaves on Spec(k) and discrete G-modules.

Proof.

We check that $M \to F_M$ is fully faithful and essentially surjective:

- $\mathrm{Hom}_G(M, M') \to \mathrm{Hom}(F_M \to F_{M'})$ is bijective, since the $F_M(k') = M^{\mathrm{Gal}(k^{\mathrm{sep}}/k')}$ cover M (discreteness condition).
- $F \cong F_{M_F}$ canonically:
  $F_{M_F}(k') = (\varinjlim F(k''))^{\mathrm{Gal}(k^{\mathrm{sep}}/k')} \cong F(k')$.
- $F \to M_F$ is also functorial, for $\phi : F \to F'$
  $\phi(k') : F(k') \to F'(k')$ commutes with the action of G, so it extends to a map on the direct limits $M_F \to M_{F'}$.

□