Étale cohomology reading seminar

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About

These are the notes from a learning seminar on étale cohomology held virtually from Oxford in the spring of 2021. They are based on my handwritten notes taken during the talks, somewhat expanded with more examples and references. As such, they are not a transcription of what the speakers said during the talks, and I have doubtlessly introduced numerous errors in the process of writing up.

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1 Introduction

Speaker: Martin Gallauer

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1.1 Why study étale cohomology?

Fix a field k, and let $f \in k[x_0, \dots, x_{n+1}]$ be a homogeneous irreducible polynomial with $(\frac{\mathrm{d}f}{\mathrm{d}x_0}, \dots \frac{\mathrm{d}f}{\mathrm{d}x_{n+1}}) \neq (0, \dots, 0)$. Then the zero-set of f over k, $X(k)$, determines a subset of $\mathbb{P}_k^{n+1}$, and very loosely speaking, the goal of algebraic geometry is to understand $X(k)$.

One powerful method of studying $X(k)$ is through its invariants. For example, if $k = \mathbb{C}$, then $X(\mathbb{C}) \subset \mathbb{P}_{\mathbb{C}}^{n+1}$ is naturally a complex manifold of real dimension $2n$, and we can define the *singular cohomology groups* $H^i(X(\mathbb{C}); \mathbb{Z})$ for $i = 0, \dots, 2n$. Then

$$H^i(X(\mathbb{C}); \mathbb{Q}) := H^i(X(\mathbb{C}); \mathbb{Z}) \otimes \mathbb{Q}$$

form $\mathbb{Q}$-vector spaces, and $b_i := \dim_{\mathbb{Q}} H^i(X(\mathbb{C}); \mathbb{Q})$ is called the *i-th Betti number* of $X(\mathbb{C})$. The *Euler characteristic* of $X(\mathbb{C})$ is defined to be $\chi(X) := \sum_{i=0}^{2n} (-1)^i b_i$.

Example 1.1. If $\deg f = 1$, then $X(\mathbb{C}) \cong \mathbb{P}_{\mathbb{C}}^n$, and one can compute that

$$b_i = \begin{cases} 1 & i \leq 2n \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

so the Euler characteristic of $X(\mathbb{C})$ is $n + 1$.

Example 1.2. If $n = 1$ and $d := \deg f \geq 2$, then the Riemann surface $X(\mathbb{C})$ has genus $g = \frac{(d-1)(d-2)}{2}$, meaning $X(\mathbb{C})$ is homeomorphic to a donut with g holes or a sphere with g handles. One can show that $(b_0, b_1, b_2) = (1, 2g, 1)$, so $\chi(X) = 2 - 2g$.

For example, if $d = 3$ then X is an elliptic curve, with genus $g = 1$ and Euler characteristic $\chi(X) = 0$.

These examples show that we can use topology to distinguish between different $X(k)$ when $k = \mathbb{C}$. However, if k is a finite field there are no such topological invariants. More precisely, if $k = \mathbb{F}_q$ where q is a prime power, then $X(k)$ is a finite set, and naively the only reasonable invariant we can define is the number of points. Let $N_r(X) := \#X(\mathbb{F}_{q^r})$ be the number of points of X defined over $\mathbb{F}_{q^r}$.

Example 1.3. If $X = \mathbb{P}_{\mathbb{F}_q}^n$, then it is straightforward (exercise!) to show that

$$N_r(\mathbb{P}_{\mathbb{F}_q}^n) = \frac{(q^r)^{n+1} - 1}{q^r - 1} = (q^r)^n + (q^r)^{n-1} \dots + q^r + 1 \tag{1.1}$$

Example 1.4. Suppose X is an elliptic curve over $\mathbb{F}_{q^r}$. Then *Hasse's theorem* gives a good estimate of each N_r:

$$|N_r(X) - q^r - 1| \leq 2q^{r/2}. \tag{1.2}$$

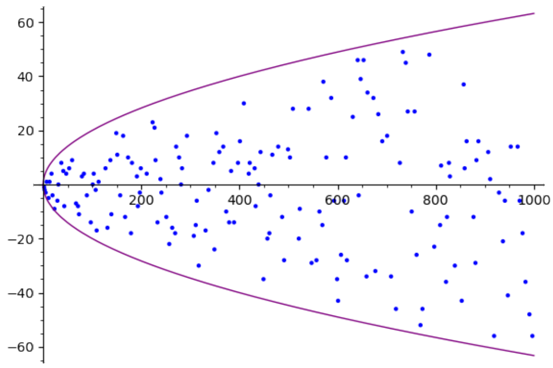


Figure 1: The value of $\#X(\mathbb{F}_p) - p - 1$ as p ranges between 1 and 1000, where X is the elliptic curve $y^2 = x^3 - 2619x + 54486$, with Hasse's bound $\pm 2\sqrt{p}$ in purple.

Weil found the following generalisation of Hasse's result:

Theorem 1.5 (Weil)**.** *Let X be a non-singular projective curve of genus g defined over $\mathbb{F}_q$. Then there exist algebraic integers $a_1, \ldots, a_{2g}$ such that:*

(i) *For every $r \geq 1$,*
$$N_r(X) = q^{r/2} + 2 - (a_1^r + \ldots + a_{2g}^r), \tag{1.3}$$

(ii) *the numbers $\{a_i\}$ are q-*Weil numbers of weight 1, *that is, $|a_i| = q^{1/2}$ for $1 \leq i \leq 2g$.*

One easily checks that this implies the Hasse-Weil theorem. Note that the property of being a q-Weil number is quite restrictive; $a_i = 3 \pm 2i\sqrt{2}$ is an example of such a number, when $q = 17$.

Taking $k = \mathbb{Q}$, let's assume furthermore that $f \in \mathbb{Z}[x_0, \ldots, x_{n+1}]$ and is primitive, and suppose the reduction mod q, $\overline{f}$ defines an irreducible and smooth $X(\mathbb{F}_q)$. For convenience, we will denote such a model of X by $\mathcal{X}$. Then we have an informal diagram

$$\begin{tikzcd} & \mathcal{X}/\mathbb{Z} \arrow[dl] \arrow[dr] & \\ \mathcal{X}(\mathbb{F}_q)/\mathbb{F}_q \arrow[rr, dashed, leftrightarrow, "?"] & & \mathcal{X}(\mathbb{C})/\mathbb{C} \end{tikzcd} \tag{1.4}$$

It is natural to ask whether there is any interaction between the structures of X over $\mathbb{F}_q$ and $\mathbb{C}$, as indicated by the arrow marked "?". For example, we might hope that there is a connection between the invariants N_r and b_i or χ. Note that the structure on the left is fundamentally arithmetic, being defined mod q, whereas the right hand side is topological.

1.2 The Weil conjectures

A satisfactory answer to this question was conjectured by Weil, and is one of the most stunning applications of étale cohomology. First we need to describe the setup:

Definition 1.6. Let $X/\mathbb{F}_q$ be as in the previous section. The *zeta function of X* is the formal power series $\zeta(X,T) \in \mathbb{Q}[[T]]$ defined by

$$\zeta(X,T) = \exp\left(\sum_{r\geq 1} \frac{N_r(X)}{r} T^r\right), \tag{1.5}$$

where $\exp(x) = \sum_{n\geq 0} x^n/n!$ is the formal exponential series.

This might look like an arbitrary definition at first, but note that $\frac{d}{dT}\log\zeta(X,T) = \sum N_{r+1}T^r$, which is the generating function of $N_r(X)$.

Example 1.7. When $X = \mathbb{P}^n_{\mathbb{F}_q}$, it is a fun exercise to show that

$$\zeta(\mathbb{P}^n,T) = \frac{1}{(1-T)(1-qT)\dots(1-q^nT)}, \tag{1.6}$$

(Hint: use eq. (1.1) and expand the resulting exponentials.)

Example 1.8. With a similar argument using Weil's theorem 1.5, one can check that if X is a curve of genus g, then

$$\zeta(X,T) = \frac{(1-a_1T)\dots(1-a_{2g}T)}{(1-T)(1-qT)}. \tag{1.7}$$

In both of the examples, we see that while $\zeta(X,T)$ is originally defined as a power series, it is actually a rational function in T defined over some finite extension of $\mathbb{Q}$. We also see that the degree of the numerator is the sum of the odd Betti numbers, while the denominator has degree the sum of the even.

Theorem 1.9 (Weil conjectures)**.** *Let $X/\mathbb{F}_q$ be a smooth projective variety of dimension n.*

(I) *$\zeta(X,T)$ is a rational function; in fact*

$$\zeta(X,T) = \frac{Q_1Q_3\cdots Q_{2n-1}}{Q_0Q_2\cdots Q_{2n}}, \tag{1.8}$$

where $Q_i \in \mathbb{Z}[T]$ are given by $Q_i := \prod_{j=1}^{b_i}(1-a_{ij}T)$ for some $b_i \in \mathbb{N}$ and algebraic integers a_{ij}.

(II) *$\zeta(X,T)$ satisfies the functional equation $\zeta(X,1/q^nT) = \pm q^{\chi n/2}T^{\chi}\zeta(X,T)$ and $\chi := \sum(-1)^i b_i$.*

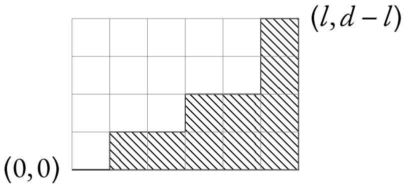
(III) The numbers a_{ij} are q-Weil numbers of weight i, meaning $|a_{ij}| = q^{i/2}$ for all $1 \le i,j \le 2n$.

(IV) If X has a "nice" model $\mathcal{X}$ defined over $\mathbb{Z}$, then b_i are precisely the Betti numbers of $\mathcal{X}(\mathbb{C})$.

(III) is referred to as the "Riemann hypothesis" for $\zeta(X,T)$, since it tells precisely where its zeroes and poles lie. Historically, the first progress on the Weil conjectures was made by Dwork who proved (I) using p-adic analytic methods. Another proof of this came with the full proof of the Weil conjectures through the work of Grothendieck and Artin, and Deligne.

While it is astonishing that information about the topology of $\mathcal{X}(\mathbb{C})$ determines the number of points when reducing modulo a prime, it is also possible to obtain information the other way, through so-called "point counting". In a sense, this is a local-to-global principle; the "global" information about the topology of $\mathcal{X}(\mathbb{C})$ is determined by "local data".

Exercise. Let $G(l,d)$ denote the complex Grassmannian (definition). Using theorem 1.9, show that $b_i(G(l,d))$ equals the number of paths on a grid from $(0,0)$ to $(l,d-l)$ with area i (where we can only move right or up).



Of course, these Betti numbers were known before the Weil conjectures were established. However, for more complicated varieties the point-counting method is sometimes one of the easiest ways of determining the topological structure.

1.3 You could have invented étale cohomology[1]

In the previous section we saw that several topological invariants of varieties over $\mathbb{C}$ defined in terms of cohomology were related to invariants over $\mathbb{F}_q$. Weil suggested that this could be possible using a "good" cohomology theory for varieties over $\mathbb{F}_q$.

Fix a topological space T and an abelian group A. We want to find a definition of cohomology groups $H^\bullet(T,A)$ which gives a meaningful answer in the context of algebraic geometry.

[1]The title is a reference to Timothy Chow's paper "You could have invented spectral sequences", see here [Cho06].

Example 1.10 (Singular cohomology)**.** Choosing $H^\bullet(T, A)$ to be singular cohomology doesn't work in general, because there are too few continuous maps in the Zariski topology.

Example 1.11 (Sheaf cohomology)**.** Let X be an irreducible scheme and $\underline{A}$ the locally constant sheaf $X \supset U \mapsto A$. Then for any pair of open sets $U, V \subset X$ with $U \subset V$ we have $\underline{A}(V) \cong A \xrightarrow{\sim} \underline{A}(U)$, and so in particular $\underline{A}$ is *flabby*, which implies that the sheaf cohomology groups $H^i(X, \underline{A})$ vanish for $i > 0$ (see [Har77, Ex. III 2.3]).

To remedy this, we first recall some sheaf theory: given a topological space T, let Op/T be the category where the objects are open subsets of T, and morphisms are given by inclusions $U \hookrightarrow V$ whenever $U \subset V$ for $U, V \in \mathrm{Op}/T$. A *presheaf* is a contravariant functor $\mathscr{F} \colon \mathrm{Op}/T \to \mathrm{Ab}$, where Ab denotes the category of abelian groups. A presheaf $\mathscr{F}$ is a *sheaf* if it satisfies the *sheaf condition*: for any $U \in \mathrm{Op}/T$ and any covering $\bigcup_i U_i$ of U with $U_i \in \mathrm{Op}/T$, we have an *equaliser diagram*:

$$\mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j). \tag{1.9}$$

Since $U_i \hookrightarrow U$, we have maps $\rho_i \colon \mathscr{F}(U) \to \mathscr{F}(U_i)$ which assemble to the first map of the diagram: $u \mapsto (\rho_i(u))_i$. The double arrows are $(u_i)_i \mapsto (\rho_{i,j}(u_i))$ and $(u_i)_i \mapsto (\rho_{i,j}(u_j))$, respectively, where $\rho_{i,j} \colon U_i \to U_i \cap U_j$. Equation (1.9) being an equaliser diagram in this case simply means that $\mathscr{F}(U)$ is the kernel of the difference of the two maps on the right.

The crucial idea is that to define sheaves, we actually don't need the full power of a topology, but rather just the notion of coverings. Regarding $U_i \cap U_j$ as the categorical fibre product $U_i \times_T U_j$, we can replace Op/T with the category of whose objects are topological spaces U equipped with a local homeomorphism $U \to T$, and where morphisms are continuous maps which factor through these. Let us denote this by $\mathrm{\acute{E}t}/T$. As before, a sheaf is any presheaf that satisfies the equaliser condition, eq. (1.9).

Proposition 1.12. *There is an equivalence of categories* $\mathrm{Sh}(\mathrm{Op}/T) \xrightarrow{\sim} \mathrm{Sh}(\mathrm{\acute{E}t}/T)$.

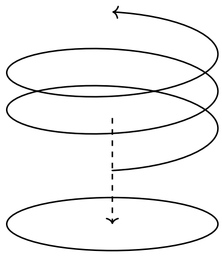
Proof. □

We can summarise this by saying that the main conceptual leap required to define étale cohomology was to replace open subsets and inclusions by coverings and local homeomorphisms.

Of course, it remains to find a good notion of a "local homeomorphism" between schemes.

Attempt 1:

Let $f: X \to Y$ be a morphism of schemes, and suppose we mimic the definition of a local homeomorphism and require that for any $x \in X$ we can find a Zariski-open neighbourhood U such that $f|_U: U \to f(U)$ is an isomorphism onto an open subscheme of Y. This is too rigid; by analogy with covering spaces of Riemann surfaces we would like the map $\mathbb{G}_m \to \mathbb{G}_m$ defined by $t \mapsto t^n$ (this corresponds to the map $z \mapsto z^n$ from $\mathbb{C}^\times$ to itself) to be a "covering map" in a suitable sense.



However, an open set in $\mathbb{G}_m$ is given by the complement of finitely many closed points, and such an open set upstairs does not look like an open downstairs.

This example also rules out the requirement that f should be an isomorphism at the stalks. Indeed, looking at the stalk at 1, we easily see that the induced map $k[t,t^{-1}]_{(t-1)} \to k[t,t^{-1}]_{(t-1)}$ given by $t \mapsto t^n$ is not surjective. The slogan here is that "stalks know too much about the global structure". However, since we are looking for a local condition, it makes sense to look for something defined in terms of stalks.

Attempt 2:

Suppose that for all closed points $x \in X$, we require the induced maps on completions $\widehat{\mathcal{O}}_{Y,f(x)} \to \widehat{\mathcal{O}}_{X,x}$ to be isomorphisms. One motivation behind this that the completion $\widehat{\mathcal{O}}_{X,x}$ "knows less than $\mathcal{O}_{X,x}$" in some sense.

Exercise. Let k be an algebraically closed field, and let $f: X \to Y$ be a morphism of smooth k-varieties (smooth separated integral finite-type k-schemes). Then the following are equivalent:

(i) For all closed points x in X, the map of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induces an isomorphism $\widehat{\mathcal{O}}_{Y,f(x)} \to \widehat{\mathcal{O}}_{X,x}$ on the completions.

(ii) For all closed points x in X, the morphism $Tf: T_xX \to T_{f(x)}Y$ on tangent spaces is an isomorphism.

(iii) If $k = \mathbb{C}$, $f: X(\mathbb{C}) \to Y(\mathbb{C})$ is a local isomorphism of smooth manifolds.

This gives rise to the notion of a morphism being *formally étale*, first coined by Grothendieck. The roadmap for developing étale cohomology is now:

(i) Develop a good theory of étale morphisms.
(ii) Develop sheaf theory in terms of covers, not opens.
(iii) Apply the two former to compute things.

Time permitting, we can look at other interesting applications.

2 Étale morphisms

Contents

2.1 Finite and quasi-finite morphisms

Speaker: Andrés Ibáñez Núñes

In what follows, all rings are assumed to be Noetherian, and all schemes locally Noetherian, meaning that they can be covered by spectra of Noetherian rings. Readers unhappy with this restriction of generality might find de Jong a more pleasing resource.

We begin by recalling the notion of a finite morphism (cf. [Har77, II.3]).

Definition 2.1. Let $f\colon X \to Y$ be a morphism of schemes. Then f is **finite** if for any affine open $V = \operatorname{Spec} B \subset Y$, the preimage $f^{-1}(V) = \operatorname{Spec} A$ is affine[2] and the induced map $B \to A$ makes A into a finitely generated B-module.

Example 2.2. All closed immersions are finite, because they locally correspond to maps of the underlying rings of the form $A \to A/I$, and A/I is a finitely generated A-module.

It is frequently useful to have a slightly weaker notion of finiteness:

Definition 2.3. A morphism of schemes $f\colon X \to Y$ is **quasi-finite** if it is of finite type[3] and if for any $y \in Y$, the preimage $f^{-1}(y)$ is a discrete topological space.

[2] That is, f is an affine morphism.

[3] i.e. for any $V = \operatorname{Spec} B \subset Y$, $f^{-1}(V)$ has a finite open affine cover $\{U_i = \operatorname{Spec} A_i\}$ such that each A_i is a finitely generated B-algebra.

In particular, this implies that the fibres are finite.

For convenience, we introduce the notion of a *stable class*:

Definition 2.4. Let $\mathscr{P}$ be a family of morphisms of schemes. $\mathscr{P}$ is a **stable class** if the following hold:

(i) $\mathscr{P}$ contains all isomorphisms;

(ii) $\mathscr{P}$ is stable under composition, meaning that if $f\colon X \to Y$ and $g\colon Y \to Z$ are members of $\mathscr{P}$, then so if $g \circ f$;

(iii) $\mathscr{P}$ is stable under base change, meaning that for any $f\colon X \to Y$ in $\mathscr{P}$, if we have a Cartesian square of the form

$$\begin{array}{ccc} Z & \xrightarrow{f'} & S \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} \tag{2.1}$$

then the morphism f' is also a member of $\mathscr{P}$.[4]

(iv) $\mathscr{P}$ is local on the target, that is, for every cover $\{V_i\}$ of Y, $f\colon X \to Y$ is in $\mathscr{P}$ if and only if the restrictions $f|_{f^{-1}(V_i)}\colon f^{-1}(V_i) \to V_i$ are in $\mathscr{P}$.

Example 2.5. The class of morphisms of finite type form a stable class, as does the classes of separated, of proper and of affine morphisms. It is an excellent exercise to list all the types of morphisms of schemes you know and decide which of the above conditions they satisfy.

Proposition 2.6. *The collection of all finite morphisms form a stable class, and so does the collection of quasi-finite morphisms.*

Proof. This is more or less routine, and omitted from the talk. Details can be found in [Mil80, Prop. 1.3]. ☐

Finite and quasi-finite morphisms into the spectrum of a field have a particularly nice interpretation.

Proposition 2.7. *Let k be a field, and $f\colon X \to \operatorname{Spec} k$ a morphism of finite type. Then the following are equivalent:*

(i) *f is finite;*

(ii) *f is quasi-finite;*

(iii) *X is affine, and $X \cong \operatorname{Spec} A$ where A is a finite dimensional k-algebra.*

[4] Normally we would simply say the "the base change $f'\colon X \times_Y S$ is in $\mathscr{P}$", but the category of locally Noetherian schemes is not closed under fibre products, see this SO post.

Proof. $(i) \Rightarrow (iii)$ follows directly from definitions.

$(iii) \Rightarrow (ii)$: Note that A is Artinian[5] because any ideal of A is k-vector space and so a strictly decreasing sequence of ideals (i.e. vector spaces) $I_0 \supset I_1 \supset \ldots$ stabilises in at most $\dim I_0$ steps. The structure theorem for Artinian rings (cf. [AM94, Thm. 8.7]) then implies that $A = \prod_{i=1}^n A_i$ where each A_i is an Artinian local ring. In particular, for each i, $\operatorname{Spec} A_i$ consists of a single point. Now $\operatorname{Spec} A = \prod_{i=1}^n A_i \cong \sqcup_{i=1}^n \operatorname{Spec} A_i$, so f is indeed quasi-finite.

$(ii) \Rightarrow (i)$ If f is quasi-finite, then since X is the preimage of the unique point of $\operatorname{Spec} k$, the underlying topological space of X is finite and discrete, and we can write $X = \sqcup_{i=1}^n \operatorname{Spec} A_i$ where each A_i is a finitely generated k-algebra and a local ring. As before, A_i is Artinian and $\operatorname{Spec} A_i$ consists of a single point, so $X = \operatorname{Spec} \prod_{i=1}^n A_i$ is the spectrum of a finite-dimensional k-algebra, so f is finite. □

The following result explains the name "quasi-finite".

Proposition 2.8. *Finite morphisms are quasi-finite.*

Proof. Let $f : X \to Y$ be finite. We know that f is of finite type, and it remains to show that the fibres are discrete. By definition of the fibre over $y \in Y$, we have a Cartesian diagram

$$\begin{tikzcd} f^{-1}(y) \arrow[r, "f'"] \arrow[d] & \operatorname{Spec} \kappa(y) \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{2.2}$$

where $\kappa(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y$ denotes the residue field of Y at y. The base change f' of f is finite by stability of finiteness under base change, and applying proposition 2.7 it is quasi-finite, hence has discrete fibres. □

It is not difficult to see that the converse is false. For example, exercise 1 of the exercise sheet ([Mil80, Ex. I.1.6b]) shows that Dedekind domains with finitely many primes are quasi-finite but never finite. Another example is the following:

Example 2.9. Fix a ring A, pick $P = a_n T^n + \ldots + a_0 \in A[T]$, set $B = A[T]/(P(T))$ and let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the natural map. Then f is finite if and only if B is a finite A-module, which one checks is equivalent to T being integral over A. But this is true if and only if the leading coefficient a_n of P is a unit.

On the other hand, f is quasi-finite if and only if for any $\mathfrak{p} \in \operatorname{Spec} A$, $B \otimes_A \kappa(\mathfrak{p}) \cong \kappa(\mathfrak{p})[T]/(P(T))$ is a finite-dimensional over $\kappa(\mathfrak{p})$. This is equivalent to requiring $P \not\equiv 0 \pmod{\mathfrak{p}}$ for all primes $\mathfrak{p}$ of A, i.e. that $(a_0, \ldots a_n) = A$. This shows that being quasi-finite is weaker than being finite, in general.

[5] Meaning any descending chain of ideals stabilises in finitely many steps.

2.2 Normalisations

Definition 2.10. A scheme X is **normal** if every stalk $\mathcal{O}_{X,x}$ is integrally closed (in its field of fractions)[6].

The notion of being normal seems to have its origins in arithmetic, and one sees that $\operatorname{Spec}\mathbb{Z}[\sqrt{5}]$ is not normal while $\operatorname{Spec}\frac{1}{2}\mathbb{Z}[\sqrt{5}]$ is. One nice property of normal schemes is that every scheme naturally admits a "normalisation":

Proposition 2.11. *Let X be an integral scheme, K the function field of X and let L/K be a field extension. Then there exists a morphism of schemes $f\colon \widetilde{X} \to X$ characterised uniquely by the following properties:*

(i) *$\widetilde{X}$ is normal,*
(ii) *f is affine,*
(iii) *for any open affine set $U \subset X$, $\mathcal{O}_{\widetilde{X}}(f^{-1}(U))$ is the integral closure of $\mathcal{O}_X(U)$ in L.*

Definition 2.12. The scheme $\widetilde{X}$ is called the **normalisation of X in L**, or simply the **normalisation of X** if $L = K$.

Normalisations give rise to a large class of finite morphisms:

Proposition 2.13 ([Mil80, Prop. I.1.1], EGA IV.7.8)**.** *Let X be a normal scheme, and $f\colon \widetilde{X} \to X$ the normalisation of X in L. If L/K is separable, or if X is of finite type over a field k, then f is finite.*

Proposition 2.14. *Let X/k be an integral scheme of finite type over a field k, with function field K. Then the normalisation $\widetilde{X} \to X$ of X in K is finite.*

Proof. We may assume X is affine, $X \cong \operatorname{Spec} A$, where A is an integral finite-dimensional k-algebra. By the Noether normalisation theorem ([AM94, Ex. 5.16]), there exists a finite injective homomorphism $k[T_1,\dots,T_n] \to A$, which extends to $k[T_1,\dots,T_n] \to \widetilde{A}$, where $\widetilde{A}$ is the integral closure of A in K. Since $\operatorname{Spec} k[T_1,\dots,T_n]$ is normal and K is a finite extension of $k(T_1,\dots,T_n)$, we have that $k[T_1,\dots,T_n] \to \widetilde{A}$ is finite by proposition 2.13, and so $A \to \widetilde{A}$ is as well. □

Example 2.15. Let k be a field and

$$A = \frac{k[x,y]}{y^2 - x^3 - x^2}, \qquad X = \operatorname{Spec} A, \tag{2.3}$$

a nodal cubic, singular at $x = 0$.

[6] Recall that this means every element of $\operatorname{Frac}(\mathcal{O}_{X,x})$ which is a root of a monic polynomial with coefficients in $\mathcal{O}_{X,x}$ must lie in $\mathcal{O}_{X,x}$.

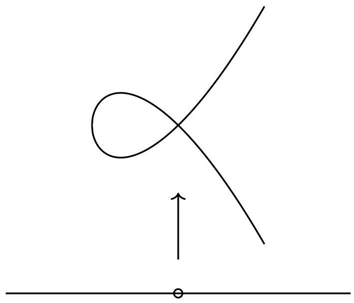


Figure 2: The nodal cubic given by $y^2 = x^3 + x^2$

Consider the map $A \to k[z]$ defined by $x \mapsto z^2 - 1$ and $y \mapsto z^3 - z$ (check by hand that this factors through quotient!). The formal computation $z^2 = y^2/x^2 = (x^3+x^2)/x^2 = x+1$ shows that $k[z]$ is integral over A. In fact, it holds that $\operatorname{Spec} k[z]$ is the normalisation of X.

Removing a single point on X corresponds to localising $k[z]$ at $(z-a)$, and we have a natural morphism $A \to k[z]_{(z-a)}$. The corresponding map of schemes is not finite, but quasi-finite, and factors as an open immersion followed by a finite morphism. This is no coincidence:

Theorem 2.16 (Zariski's main theorem)**.** *Let Y be quasi-compact, and $f : X \to Y$ a separated and quasi-finite morphism. Then f factors as $X \xrightarrow{i} X' \xrightarrow{g} Y$, where i is an open immersion and g a finite morphism.*

Remark. The condition that f be separated is necessary, since a finite morphism is affine, hence separated.

Proof. See [Mil80, Thm. I.1.8]. If we additionally assume that f is projective, then it is possible to deduce this from the Zariski main theorem in Hartshorne's book, [Har77, Cor. III.11.4]. $\square$

This is a different version of Zariski's main theorem from say, the one in [Har77]. For a nice overview of results going by the name "Zariski's main theorem", see [Mum67, Sec. III.9].

We end this section with a useful characterisation of finite morphisms:

Proposition 2.17. *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

(i) *f is finite,*
(ii) *f is proper and quasi-finite,*
(iii) *f is proper and affine.*

Proof. We first prove the equivalence $(i) \Leftrightarrow (ii)$:

$(i) \Rightarrow (ii)$ We already know finite morphisms are quasi-finite, so it remains to prove properness. Recall that being proper means being separated, of finite type and universally closed[7]. Finite morphisms are affine, hence separated ([Har77, Ex. II.5.17b]) and of finite type, so it remains to show that they are universally closed. Since being finite is stable under base change, it suffices to show that f is closed. We reduce further to requiring $f(X)$ to be closed as follows:

If we know that $f(X)$ is closed for all finite morphisms f, then for any closed set $Z \subset X$ we have a closed immersion $Z \to X$, the composition $Z \to X \xrightarrow{f} Y$ is finite, so $f(Z)$ is closed.

In this case we can reduce to the case where Y (and hence X) is affine, since closedness can be checked locally. Then f factors as $X = \operatorname{Spec} A \xrightarrow{u} \operatorname{Spec} B/I \xrightarrow{v} \operatorname{Spec} B = Y$, where u is surjective by the lying-above theorem [AM94, Thm. 5.10], and v is a closed immersion. It follows that $f(X)$ is closed.

$(ii) \Rightarrow (i)$ Since finiteness is local on the target, we can assume that Y is quasi-compact, so by Zariski's main theorem we can factor f as $X \xrightarrow{u} X' \xrightarrow{g} Y$ where u is an open immersion. We claim that u is proper; from this it will follow that u is a closed immersion, so f is a composition of finite morphisms, hence itself finite.

Indeed, let's write u as the composition $X \xrightarrow{(\mathrm{Id}_X, u)} X \times_Y X' \xrightarrow{\mathrm{pr}_2} X'$, where the fibre product is taken over f. Then pr_2 is proper, being the base change of f, and we claim that (Id_X, u) is also proper. Indeed, we have a Cartesian diagram

$$\begin{tikzcd} X \arrow[r, "(\mathrm{Id}_X, u)"] \arrow[d] & X \times_Y X' \arrow[d] \\ X' \arrow[r, "\Delta_g"] & X' \times_Y X' \end{tikzcd} \tag{2.4}$$

which shows that (Id_X, u) is the base change of the diagonal morphism Δ_g, which is a closed immersion since g is separated. Being a closed immersion is stable under base change, so we conclude that (Id_X, u) is indeed a closed immersion, and this proves (i).

$(i) \Rightarrow (iii)$ is now clear using (ii), and the converse follows from finiteness-theorems of proper morphisms, see for example EGA II, 6.7.1. $\square$

2.3 Flat morphisms

Mumford eloquently describes flatness as "a riddle that comes out of algebra, but which technically is the answer to many prayers" [Mum67, Sec. III.10]. One of the solutions he offers is that a flat morphism "preserves linear structure", and in

[7] Any base change of f is closed.

a continuously varying family of schemes we can recognise it by the statement that the dimension of fibres remains constant as the parameter varies.

In the following, we adopt the convention of denoting a short exact sequence of A-modules $0 \to M' \to M \to M'' \to 0$ by Σ, and if N is another A-module, let $\Sigma \otimes_A N$ denote the sequence

$$0 \to M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0. \tag{2.5}$$

Definition 2.18. A map of rings $\phi : A \to B$ is **flat** if for every short exact sequence Σ of A-modules, $\Sigma \otimes_A B$ is exact. A morphism of schemes $f : X \to Y$ is **flat** if for every $x \in X$, the corresponding map of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

Of course, one should check that these notions are compatible if X and Y are affine:

Proposition 2.19. *A morphism of rings $\phi : A \to B$ is flat if and only if the corresponding map* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is flat.*

Proof. This can be rephrased as saying that being flat is a local property, and this is the content of [AM94, Prop. 2.19]. □

Proposition 2.20. *Flat morphisms form a stable class.*

Proof. This is mostly straightforward checking. □

Example 2.21. Open immersions are flat, since each map of stalks is simply the identity.

Another class of morphisms that shows up frequently in scheme theory is the following:

Definition 2.22. A morphism $f : X \to Y$ is **faithfully flat** if it is flat and surjective.

Proposition 2.23. *For a flat morphism of rings $\phi : A \to B$, the following are equivalent:*

(i) *For every A-module M, if $M \neq 0$ then $M \otimes_A B \neq 0$,*
(ii) *for every sequence $\Sigma = 0 \to M' \to M \to M''$ of A-modules, exactness of $\Sigma \otimes_A B$ implies the exactness of Σ.*
(iii) *the associated morphism* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is faithfully flat,*
(iv) *for every maximal ideal $\mathfrak{m} \subset A$, $\phi(\mathfrak{m})B$ is a strict subset of B.*

Checking condition (iv) immediately gives the following:

Corollary 2.24. *A local homomorphism of local rings is faithfully flat.*

Corollary 2.25. *If $f : X \to Y$ is a flat morphism, then $f(X)$ is "closed under generalisation". In other words, if $f(x) \in \overline{\{y\}}$ for some $x \in X$, then $y = f(x')$ for some $x' \in X$.*

Proof. We have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_{X,x} \arrow[r, "\ell"] \arrow[d, hook] & \operatorname{Spec} \mathcal{O}_{Y,f(x)} \arrow[d, hook] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{2.6}$$

We can identify $\operatorname{Spec} \mathcal{O}_{Y,f(x)}$ with the set of generalisations of $f(x)$. By corollary 2.24, the map ℓ is faithfully flat hence surjective, so if $f(x)$ is in the closure of y, then we can find $x' \in \operatorname{Spec} \mathcal{O}_{X,x}$, which we can identify with a generalisation of x. Commutativity of the diagram then implies $f(x') = y$. □

The goal of setting up this machinery is to prove the following important theorem:

Theorem 2.26 ([Mil80],Thm. 1.1.8)**.** *If a morphism $f : X \to Y$ of schemes is flat and locally of finite type, then it is open.*

Proof. To prove this, we will require Chevalley's theorem:

Theorem (Chevalley)**.** *Let $f: X \to Y$ be a morphism of finite type between Noetherian schemes. If $E \subset X$ is constructible*[8]*, then $f(X) \subset Y$ is also constructible.*

We will not prove this here, but one reference is EGA IV, Thm. 1.8.4.

Assume that Y is quasi-compact (hence Noetherian), and that f is of finite type. Flatness being local on the source (exercise!), it suffices to show that $f(X)$ is open, and Chevalley's theorem then implies $f(X)$ is open. The result will then follow from corollary 2.25 and the following lemma:

Lemma 2.27. *Let Y be a Noetherian scheme and let $S \subset Y$ be a subset. Then S is open if and only if S is constructible and stable under generalisation.*

Details to be filled out. □

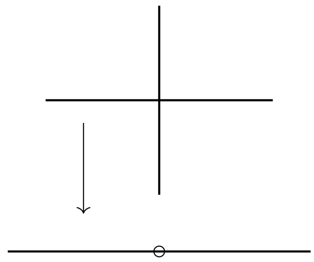
2.4 Unramified morphisms

Speaker: Håvard Damm-Johnsen

[8]Recall that E is *constructible* if it is a finite union of locally closed subsets.

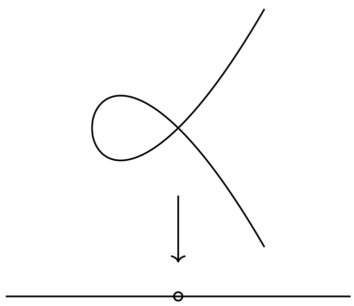
Recall that we are trying to find a nice notion of local isomorphism for schemes. By local, we mean that it should be defined in terms of stalks, and it is desirable to find a notion that holds over arbitrary rings, not just fields.

Example 2.28. Consider the affine scheme $X = \operatorname{Spec} A$ where $A = k[x,y]/(xy)$, regarded as a scheme over $\operatorname{Spec} k[x]$, and we denote by $f : X \to \mathbb{A}^1_k$ the associated morphism. Geometrically, this is a cross along with the projection onto the x-axis.



Heuristically, f is not flat because of the "jump in dimension" of the fibre at 0 compared to the nearby fibres. In formal terms, note that A is a PID, so flatness is equivalent to being torsion-free [LE06, Cor. 1.2.5]. But the localisation $A_{(x,y)}$ viewed as a $k[x]_{(x)}$-module has torsion because $xy = 0$. This demonstrates that flatness should be a necessary condition for being étale. On the other hand, the fibre of (x) is the only place of non-flatness for f, so it should be étale elsewhere.

Example 2.29. Let's return to the nodal cubic in example 2.15, this time with a projection f onto the x-axis.



This corresponds to the natural map $k[x] \to k[x,y]/(y^2 - x^3 - x^2)$, and intuitively f should not be a local isomorphism at the singularity $(0,0)$, because locally there are "four branches" coming out of the point. Readers familiar with Riemann surfaces might recognise this as a ramification point, in the context of which f locally looks like $z \mapsto z^2$ near $(0,0)$.

For[9] a map $g\colon X \to Y$ of Riemann surfaces we have an associated map $\mathscr{M}(Y) \to \mathscr{M}(X)$ of meromorphic function fields, and we can look at the subrings $\mathscr{O}_x \subset \mathscr{M}(X)$, $\mathscr{O}_{g(x)} \subset \mathscr{M}(Y)$ of functions holomorphic at x and $g(x)$, respectively, and g induces a map $\mathscr{O}_{g(x)} \to \mathscr{O}_x$. Here $\mathfrak{m}_{g(x)} \subset \mathscr{O}_{g(x)}$, the ideal of functions vanishing at x, is mapped into the corresponding ideal $\mathfrak{m}_x \subset \mathscr{O}_x$. We see that this is a map of local rings, completely analogous to the map of stalks for f. Identifying the ideal with its image, we see that $\mathfrak{m}_{g(x)}\mathscr{O}_x = \mathfrak{m}_x^{e_x}$, where e_x is the *ramification index*, and corresponds to the "number of branches of g". In this setting, g is said to be *unramified at x* if $e_x = 1$. This transfers almost verbatim to schemes, with the additional requirement of separability.

Definition 2.30. A morphism $f\colon X \to Y$ of schemes locally of finite type is **unramified at** $x \in X$ if the following two conditions hold:

(i) $\mathfrak{m}_{f(x)}$ generates the maximal ideal of $\mathscr{O}_{X,x}$, that is, $\mathfrak{m}_{f(x)}\mathscr{O}_{X,x} = \mathfrak{m}_x$.
(ii) The corresponding field extension $\kappa(x)/\kappa(f(x))$, where $\kappa(x) := \mathscr{O}_{X,x}/\mathfrak{m}_x$ and $\kappa(f(x)) := \mathscr{O}_{Y,f(x)}/\mathfrak{m}_{f(x)}$, is separable.

If f is unramified at all $x \in X$, we simply say it is **unramified**.

This definition is sometimes a bit unwieldy; fortunately the following makes computations easier in practice.

Proposition 2.31. *Let $f : X \to Y$ be a morphism locally of finite type. The following are equivalent:*

(i) *f is unramified at x;*
(ii) $(\Omega_{X/Y})_x = 0$*;*
(iii) *The diagonal morphism $\Delta_{X/Y}$ is an open immersion.*

Proof. $(iii) \Rightarrow (i)$ is somewhat tedious, and we refer the eager reader to [Mil80, Prop. I.3.5].

$(i) \Rightarrow (ii)$: The question is local, so assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Let $\mathfrak{p} = x$, $\mathfrak{q} = f(x)$. Then we have a map $\phi\colon B_\mathfrak{q} \to A_\mathfrak{p}$ which by hypothesis satisfies $\phi(\mathfrak{q})A_\mathfrak{p} = \mathfrak{p}$. It follows that $A_\mathfrak{p} \otimes_{B_\mathfrak{q}} \kappa(\mathfrak{q}) \cong \kappa(\mathfrak{p})$, so we have a Cartesian diagram

$$\begin{tikzcd} \operatorname{Spec}\kappa(\mathfrak{p}) \arrow[r] \arrow[d] & \operatorname{Spec} A_\mathfrak{p} \arrow[d] \\ \operatorname{Spec}\kappa(\mathfrak{q}) \arrow[r] & \operatorname{Spec} B_\mathfrak{q} \end{tikzcd} \tag{2.7}$$

Now [Har77, Prop. II.8.2A] implies that $\Omega_{A_\mathfrak{p}/B_\mathfrak{q}} \otimes_{A_\mathfrak{p}} \kappa(\mathfrak{p}) = \Omega_{\kappa(\mathfrak{p})/\kappa(\mathfrak{q})}$, which vanishes identically by the hypothesis (see [Mum67, p.283] for a hint). Therefore,

[9] Thanks to George for bringing up this analogy!

by Nakayama's lemma, $\Omega_{A_\mathfrak{p}/B_\mathfrak{q}} = 0$ since it is finitely generated over $A_\mathfrak{p}$. Next we have "the first exact sequence"

$$\Omega_{B_\mathfrak{q}/B} \otimes B_\mathfrak{q} A_\mathfrak{p} \to \Omega_{A_\mathfrak{p}/B} \to \Omega_{A_\mathfrak{p}/B_\mathfrak{q}} = 0 \to 0, \tag{2.8}$$

by [Har77, Prop. II.8.3A], which implies that $0 = \Omega_{A_\mathfrak{p}/B} = (\Omega_{X/Y})_x$, which proves our claim.

$(ii) \Rightarrow (iii)$ As in the previous part we assume that X and Y are affine, and in this case $\Delta_{X/Y}$ is the map of schemes associated to $m\colon A \otimes_B A \to A$, defined by $m(a \otimes a') = aa'$. Note that m is surjective since $\Delta_{X/Y}$ is a closed immersion, as we are in an affine setting. Note that [Har77, Prop. II.8.1A], $\Omega_{X/Y} = I/I^2$ where $I = \ker m$, so by hypothesis $0 = (\Omega_{X/Y})_x = (I/I^2)_{\Delta(\mathfrak{p})} \cong I_{\Delta(\mathfrak{p})}/I^2_{\Delta(\mathfrak{p})}$. Since f is of finite type, we can apply Nakayama's lemma to deduce that $I_{\Delta(\mathfrak{p})} = 0$. Now by exercise 13.7.E in Vakil's notes, I vanishes in a neighbourhood of U of $\Delta(x)$, so $\Delta|_{\Delta^{-1}(U)}\colon \Delta^{-1}(U) \to U$ is an isomorphism and in particular an open immersion.

□

Returning to the cubic in example 2.29, we compute the sheaf of relative differentials

$$\Omega_{A/k[x]} = \frac{A dx + A dy}{(2y dy - (3x^2 + 2x) dx) A}, \tag{2.9}$$

and one easily checks that the localisation at a prime $\mathfrak{p} \in \operatorname{Spec} A$ is identically 0 if and only if $\mathfrak{p} \neq (x, y)$.

An easy consequence of the above criteria for being unramified is the following:

Proposition 2.32. *Unramified morphisms form a stable class.*

The notion of ramification of schemes also extends the corresponding notion in number theory, as the following example indicates:

Example 2.33. Recall that the prime elements of $\mathbb{Z}[i]$ are given by

(i) primes $p \in \mathbb{Z}$ where $p \equiv 3 \pmod 4$,
(ii) $n + mi$ if $p := n^2 + m^2$ is a prime with $p \equiv 1 \pmod 4$,
(iii) $1 + i$.

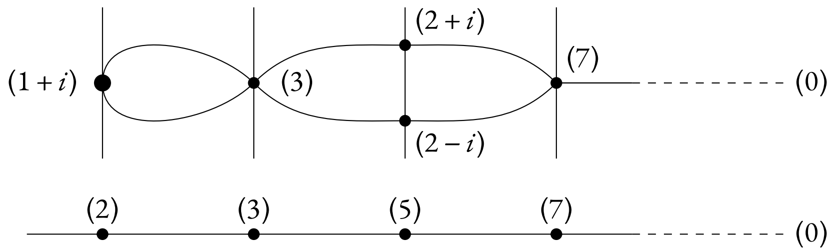
(see e.g. [NS13, Thm. 1.4.]) To study the geometry of $\operatorname{Spec} \mathbb{Z}[i]$, let us consider the fibres under the canonical map f into $\operatorname{Spec} \mathbb{Z}$. Fix a prime $(p) \in \operatorname{Spec} \mathbb{Z}$. Then

$$\operatorname{Spec} \mathbb{Z}[i] \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \kappa(p) = \operatorname{Spec}(\mathbb{Z}[i] \otimes_\mathbb{Z} \mathbb{F}_p) = \operatorname{Spec} \mathbb{F}_p[i],$$

and consider first the case where $p = 2$. Since $\mathbb{F}_p[i] \cong \mathbb{F}_p[x]/(x^2+1)$, this ring has four elements. But via the automorphism $x \mapsto x + 1$, we see that $\mathbb{F}_2[i] \cong \mathbb{F}[x]/x^2$, so the fibre of 2, which consists of only the point $(1+i)$, is a *fat point*, since the fibre is not a field.

Taking $p \equiv 3 \pmod 4$, we claim that the fibre of (p) is a field. Indeed, x^2+1 is irreducible in $\mathbb{F}_p[x]$, hence generates a maximal ideal, so $\mathbb{F}_p[x]/(x^2+1) \cong \mathbb{F}_{p^2}$. On the other hand, if $p \equiv 1 \pmod 4$, then x^2+1 is not irreducible over $\mathbb{F}_p$, but decomposes as the product of two linear factors $P_1(x)$ and $P_2(x)$. Then we have a corresponding decomposition of the fibre, as $\mathbb{F}_p[x]/P_1(x) \times \mathbb{F}_p[x]/P_2(x) \cong \mathbb{F}_p \times \mathbb{F}_p$.

We can draw the picture as follows:



Looking at the local rings, we see that $\mathbb{Z}_{(2)} \to \mathbb{Z}[i]_{(i+1)}$ sends (2) to $(1+i)^2$, so f is ramified at 2.

Exercise. Using the fact that $\mathbb{Z}[\sqrt{d}]$ is ramified precisely at primes dividing the discriminant $4d$, try to draw pictures of $\operatorname{Spec} \mathbb{Z}[\sqrt{d}]$ for some squarefree $d \in \mathbb{Z}$, including composite numbers.

2.5 Étale morphisms

The previous section hopefully convinced you that being flat and unramified are necessary conditions for being a local isomorphism. It turns out that they are also sufficient!

Proposition 2.34 (EGA IV 17.6.3)**.** *Let $f : X \to Y$ be locally of finite type. Suppose $x \in X$ satisfies $\kappa(x) \cong \kappa(f(x))$. Then f is flat at x and unramified at x if and only if the induced map $\widehat{\mathcal{O}}_{Y,f(x)} \to \widehat{\mathcal{O}}_{X,x}$ is an isomorphism.*

Definition 2.35. A morphism $f : X \to Y$ locally of finite type is **étale at** $x \in X$ if it is flat at x and unramified at x. If it is étale at every $x \in X$, we simply say that f **is étale**.

An immediate consequence of proposition 2.20 and proposition 2.32 is the following:

Proposition 2.36. *Étale morphisms form a stable class.*

Example 2.37. The nodal cubic and cross of examples 2.28 and 2.29 respectively, are étale on the complements of the problematic points.

Example 2.38. Fix a Noetherian ring A and $P(x) \in A[x]$. It is natural to ask when the morphism $\operatorname{Spec} A[x]/(P(x)) \to \operatorname{Spec} A$ is étale. It is easy to see that a sufficient condition for flatness is that P be monic, and in general, it turns out that flatness is equivalent to the statement that the ideal of A generated by the coefficients of P is generated by an idempotent.

To be unramified, we recall from Galois theory that a necessary and sufficient condition is that $P(x)$ is separable, that is, has no repeated roots. This is equivalent to the statement that $(P(x), P'(x)) = 1$, where $P'(x)$ is the formal derivative of P, and we can rephrase this as saying that $P'(x) \in (A[x]/P(x))^\times$. Here the map $\operatorname{Spec} A[x]/(P(x)) \to \operatorname{Spec} A$ is a special case of what we call a *standard étale morphism*.

Definition 2.39. Let A be a Noetherian ring, $P(x) \in A[x]$ be a monic polynomial, $B := A[x]/(P(x))$ and fix $b \in B$ such that $P'(x) \in B[b^{-1}]^\times$. A **standard étale morphism** is a morphism of the form $\operatorname{Spec} B[b^{-1}] \to \operatorname{Spec} A$.

The reason for the name is that all étale morphisms locally look like standard étale morphisms.

Theorem 2.40. *Let $f : X \to Y$ be a morphism locally of finite type. Then f is étale at $x \in X$ if and only if there exist affine open neighbourhoods U containing x and V containing $f(x)$ such that $f|_U : U \to V$ is a standard étale morphism.*

Proof. See [Mil80, Thm. 1.3.14], or [Sta21, Section 02GH] for a slightly more modern treatment. □

Corollary 2.41. *A morphism $f : X \to Y$ locally of finite type is étale at $x \in X$ if and only if there exist affine open neighbourhoods $U \cong \operatorname{Spec} R$ containing x and $V \cong \operatorname{Spec} S$ containing $f(x)$ such that*

$$R \cong \frac{S[T_1, \dots, T_n]}{(P_1, \dots, P_n)} \quad \textit{and} \quad \det\left(\frac{\partial P_i(T_1, \dots, T_n)}{\partial T_j}\right)_{i,j} \in R^\times \tag{2.10}$$

This is frequently referred to as the "Jacobian criterion" for étale morphisms, and should be seen as an analogue of the implicit function theorem from differential geometry.

Proof. $\Leftarrow$ To show that f is unramified at x, it suffices to show that $(\Omega_{R/S})_x = 0$. By definition,

$$\Omega_{R/S} = \frac{\langle dT_1, \dots, dT_n \rangle_R}{\left\langle \frac{\partial P_i}{\partial T_1} dT_1 + \dots + \frac{\partial P_i}{\partial T_n} dT_n : i = 1, \dots, n \right\rangle_R}, \tag{2.11}$$

and since $\det\left(\frac{\partial P_i(T_1, \dots, T_n)}{\partial T_j}\right)_{i,j} \in R^\times$, the quotient is related to $dT_1, \dots, dT_n$ by the linear transformation corresponding to the Jacobian matrix. Since this is invertible, we are quotienting by everything, and in particular the stalk at x vanishes identically.

Flatness at x follows from an argument similar to (but slightly more involved than) the example above, see [Mum67, Thm. III.10.3'] for more details.

$\Rightarrow$ By theorem 2.40, we can find affine open neighbourhoods $U \cong \operatorname{Spec} R$ and $V \cong \operatorname{Spec} S$ such that $R \cong \left(\frac{S[T]}{P(T)}\right)[b^{-1}]$ for some $b \in \frac{S[T]}{P(T)}$ such that $P'(T) \in R^\times$. Now note that b^{-1} is a zero of the polynomial $bU - 1 \in \frac{S[T]}{P(T)}[U]$, and so

$$R = \frac{S[T,U]}{(P(T), bU-1)}. \tag{2.12}$$

It remains to check that the corresponding Jacobian matrix is invertible. But

$$\det\begin{pmatrix} \frac{\partial P(T)}{\partial T} & \frac{\partial P(T)}{\partial U} = 0 \\ \frac{\partial (bU-1)}{\partial T} & \frac{\partial (bU-1)}{\partial U} \end{pmatrix} = P'(T)\cdot b, \tag{2.13}$$

which is in $R^\times$ by assumption. This actually proves the slightly stronger statement that we can take $n = 2$. □

An easy consequence of this is the following:

Corollary 2.42. *Let $f : X \to Y$ be étale. Then*

(i) $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,f(x)}$ *for all* $x \in X$;
(ii) *if Y is normal, then X is normal;*
(iii) *if Y is regular, then X is regular.*

Recall from the guiding examples in the beginning of this section that on the complement of a closed set, f was étale. This is no accident; the "problematic points" always form a closed set, as the following proposition shows.

Proposition 2.43. *Let $f : X \to Y$ be locally of finite type. Then the étale locus, meaning the set of points $x \in X$ at which f is étale, is an open set.*

Proof. Evidently the étale locus is the intersection of the set of flat points and the set of unramified points. The flat locus is open by commutative algebra (see EGA IV Thm. 11.3.1 or [MR89, §24]), and the unramified locus is open since it is cut out by the *different ideal sheaf*, as explained in the exercises for this week. □

A useful result for later is the following:

Proposition 2.44. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes such that gf is étale and g is unramified. Then f is étale.*

Proof. We apply the trick of factoring f as $\mathrm{pr}_2 \Gamma_f$ from proposition 2.17. Recall that the graph morphism Γ_f is defined as the base change of the diagonal morphism Δ_g,

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"] & X \times_Z Y \arrow[r, "\mathrm{pr}_2"] \arrow[d] & Y \\ Y \arrow[r, "\Delta_g"] & Y \times_Z Y & \end{tikzcd}$$

and since g is unramified, proposition 2.31 implies that Δ_g is an open immersion, hence étale. Now Γ_f is étale, since étale is stable under base change.

Similarly, the morphism pr_2 arises from the usual Cartesian diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\mathrm{pr}_2"] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "gf"] & Z \end{tikzcd}$$

and since the composition gf is étale, so is its base change pr_2. Since being étale is stable under composition, this proves our result. □

3 Étale sheaves

Contents

3.1 Sites and Grothendieck topologies

Speaker: Martin Ortiz Ramirez

Recall from section 1.3 that for the purpose of defining sheaves, we don't need a full topology but rather a notion of open *covers*. In the context of schemes, we do this by viewing an open subset $U \subset X$ as an open immersion $U \hookrightarrow X$, $U \cap V$ as $U \times_X V$, and so on. The axioms required to define a sheaf turn out to be the following:

Definition 3.1. Let C be a category. A **Grothendieck topology** T **on** C consists of collections of distinguished maps $\{U_i \to U\}_{i\in\mathcal{J}}$, *coverings of* U, for each $U \in \mathrm{C}$, satisfying the following axioms:[10]

(i) If $U_i \to U$ and $U_j \to U$ are coverings, then $U_i \times_U U_j \to U$ is also a covering.
(ii) If $\{U_i \to U\}_{i\in\mathcal{J}}$ and $\{U_{ij} \to U_i\}_{j\in\mathcal{J}}$ are coverings of U and U_i respectively, then $\{U_{ij} \to U\}_{(i,j)\in\mathcal{J}\times\mathcal{J}}$ is a covering of U.
(iii) The set consisting of the identity map $\{U \to U\}$ is a covering.

We call the pair (C, T) a **site**.

Example 3.2. If X is any topological space, then the category $\mathrm{U}(X)$ of open subsets where arrows are given by inclusions forms a site, with coverings are given by collections of open inclusions $\{\phi_i : U_i \to U\}_{i\in\mathcal{J}}$ such that $\bigcup_i \phi_i(U_i) = U$ ("surjective families").

If X is a scheme, then this is called the **Zariski site**, denoted by X_{Zar}.

Example 3.3. The **small étale site** on a scheme X, $X_{\text{ét}}$, is the category of étale X-schemes $Y \to X$. Note that if $Y \to X$ and $Z \to X$ are étale X-schemes and $Y \to Z$ is a morphism of X-schemes, that is, a morphism of schemes such that the diagram

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Z \arrow[dl] \\ & X & \end{tikzcd} \tag{3.1}$$

commutes, then proposition 2.44 implies that $Y \to Z$ is also étale.

Since every open immersion is étale, there is a natural inclusion of X_{Zar} into $X_{\text{ét}}$.

Recall that a *presheaf* on a category C with values in C′ is a contravariant functor $\mathrm{C} \to \mathrm{C}'$, and a morphisms of presheaves is simply a natural transformation of functors.

Definition 3.4. A **sheaf on a site** T is a presheaf $\mathcal{F} : \mathrm{C} \to \mathrm{C}'$ such that for all coverings $\{U_i \to U\}_{i\in\mathcal{J}}$, the diagram

$$\mathcal{F}(U) \to \prod_{i\in\mathcal{J}} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in\mathcal{J}\times\mathcal{J}} \mathcal{F}(U_i \times_U U_j) \tag{Sh}$$

is an equaliser diagram, which was defined after eq. (1.9) in the introduction. We will refer to this as the *sheaf condition*.

A **morphism of sheaves on** T is a morphism of presheaves, that is, a natural transformation.

[10] Note: the name "Grothendieck pre-topology" is frequently found in the literature. See this for an explanation of the differences.

As with sheaves on a topological space, we refer to the maps $\mathcal{F}(\phi)$ as *restriction maps*, and the corresponding category is denoted by $\mathrm{Sh}(\mathrm{T})$. Unless specified, we assume that $\mathrm{C}' = \mathrm{Ab}$, the category of abelian groups.

Definition 3.5. A sheaf on the small étale site $X_{\text{ét}}$ is called an **étale sheaf**.

Note that every étale sheaf is necessarily also a sheaf for the Zariski site, which is the same as a sheaf in the traditional sense. While it is not always easy to check if a presheaf is a sheaf, the following proposition gives a useful criterion:

Proposition 3.6. *Let $\mathcal{F}$ be a presheaf on the category of étale X-schemes. Then $\mathcal{F}$ is an étale sheaf if and only if it is a sheaf on the Zariski site and for any covering $V \to U$ of affine étale X-schemes, the following is an equaliser diagram:*

$$\mathcal{F}(U) \to \mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_U V). \tag{3.2}$$

In other words, we need only check on affine étale coverings consisting of a single map.

Proof. □

Example 3.7. Given an étale map $U \to X$, define $\mathcal{O}_{X_{\text{ét}}}(U) = \Gamma(U, \mathcal{O}_U)$. This is a Zariski sheaf because it coincides with the structure sheaf when $U \hookrightarrow X$ is an open immersion, and we want to show that it is an étale sheaf by checking the criterion above. If $\operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of X-schemes, then we need to check that

$$A \to B \rightrightarrows B \otimes_A B \tag{3.3}$$

is an equaliser diagram. Here the double arrow corresponds to $b \mapsto b \otimes 1$ and $b \mapsto 1 \otimes b$. Since the category of rings is additive, this is equivalent to exactness of

$$0 \to A \to B \xrightarrow{b \mapsto b \otimes 1 - 1 \otimes b} B \otimes_A B, \tag{3.4}$$

which follows from the fact that $\operatorname{Spec} A \to \operatorname{Spec} B$ is faithfully flat.

Example 3.8. Let Z be an X-scheme, and consider the presheaf $U \mapsto \operatorname{Hom}_X(U, Z)$. It is not difficult to check that this is in fact a sheaf on X_{Zar}, and we claim that it is an étale sheaf. For affine $Z \cong \operatorname{Spec} R$, the exactness of the equaliser diagram

$$Z(A) \to Z(B) \rightrightarrows Z(B \otimes_A B) \tag{3.5}$$

follows from exactness of eq. (3.4), since the associated diagram of rings is

$$\operatorname{Hom}(R, A) \to \operatorname{Hom}(R, B) \rightrightarrows \operatorname{Hom}(R, B \otimes_A B). \tag{3.6}$$

This extends to not necessarily affine Z through a standard patching argument.

For a concrete example, taking

$$Z = \operatorname{Spec} \frac{\mathbb{Z}[t, t^{-1}]}{(t^n - 1)} \times_{\operatorname{Spec} \mathbb{Z}} X$$

we obtain μ_n, which is the usual group scheme with $\mu_n(U) = \ker(\Gamma(U, \mathcal{O}_U) \xrightarrow{s \mapsto s^n} \Gamma(U, \mathcal{O}_U))$ of n-th roots of unity.

Example 3.9. Let X be a quasi-compact scheme, A an abelian group, and let $\underline{A}$ denote the presheaf which sends U to the set of functions $U \to A$ which are constant on each connected component. We recognise this as the sheafification of the constant presheaf $U \mapsto A$.[11] The sheaf $\underline{A}$ is called the **constant sheaf** associated to A.

Example 3.10. Anologously to the Zariski case, we can define a **locally constant sheaf** $\mathcal{F}$ for the étale topology by requiring that for some covering $\{U_i \to U\}_{i \in \mathcal{I}}$, $\mathcal{F}|_{U_i}$ is constant for all $i \in \mathcal{I}$. We will see an example of a locally constant non-constant étale sheaf in the next section.

3.2 Étale sheaves over a field

Let G be a group. By a G-module, we mean a module of the associated group ring, $\mathbb{Z}[G]$ which consists of finite formal sums of elements of g, with multiplication given by the group operation. If G is a compact topological group, then we say a G-module M is *discrete* if the stabiliser of each element of M is an open subgroup of G. This is equivalent to endowing M with the discrete topology and requiring the action of G to be continuous.

Example 3.11. If k is a field, then we can consider a *separable closure* k^{sep}, which by definition is the union of all finite separable extensions of k inside a fixed algebraic closure k^{alg}. It is not difficult to show that k^{sep} is a Galois extension, and we let $G \coloneqq \mathrm{Gal}(k^{\mathrm{sep}}/k)$.

G is an example of a *profinite group*, a topological group isomorphic to an inverse limit of finite groups viewed as discrete topological groups: a fundamental result in Galois theory states that $\mathrm{Gal}(k^{\mathrm{sep}}/k) = \varprojlim \mathrm{Gal}(L/k)$, where L runs over finite Galois extensions of k. Moreover, any subextension of k^{sep} is naturally a discrete G-module.

In this section, the goal is to prove the following theorem:

Theorem 3.12 ([Mil80, Thm. II.1.9])**.** *Let k be a field, k^{sep} a fixed separable closure, and $G \coloneqq \mathrm{Gal}(k^{\mathrm{sep}}/k)$. There is an equivalence of categories between the category of étale sheaves on $\mathrm{Spec}\, k$ and the category of discrete G-modules.*

To prove this, it is convenient to introduce the notion of an *étale algebra over k*, which is a finite product of finite separable extensions of k. A ring A is an étale algebra if and only if the map $\mathrm{Spec}\, A \to \mathrm{Spec}\, k$ is étale. Étale k-algebras form a category $\mathrm{Alg}_{\text{ét}}(k)$ with morphisms given by k-algebra maps.

If $\mathcal{F}$ is a presheaf on $X_{\text{ét}}$ where $X = \mathrm{Spec}\, k$, then by composing with the functor Spec we can naturally identify $\mathcal{F}$ with a *covariant* functor $\mathrm{Alg}_{\text{ét}}(k) \to \mathrm{Ab}$.

[11] We have not proved this, but there is a sheafification functor on the category of étale sheaves. See [Sta21, Section 00W1] for more details.

Lemma 3.13. *With notation as above, a presheaf $\mathscr{F}$ is an étale sheaf if and only if the following two conditions hold:*

(i) $\mathscr{F}(\prod A_i) = \bigoplus \mathscr{F}(A_i)$ *for all finite sets of étale algebras* $\{A_i\}$;
(ii) *for all finite Galois extensions L'/L with L/k a finite separable extension, the fixed set of $\mathscr{F}(L')$ under the action of* $\mathrm{Gal}(L'/L)$ *equals* $\mathscr{F}(L)$.

Explicitly, $\mathrm{Gal}(L'/L)$ acts on $\mathscr{F}(L)$ by $(\sigma, x) \mapsto \mathscr{F}(\sigma)(x)$ for $x \in \mathscr{F}(L)$.

Proof. In light of proposition 3.6, $\mathscr{F}$ is a Zariski sheaf if and only $\mathscr{F}(\prod A_i) = \bigoplus \mathscr{F}(A_i)$ for all étale algebras A_i because any $U \to \operatorname{Spec} k$ is discrete. If this holds, then by passing to restrictions we see that $\mathscr{F}$ is étale if and only if for any pair L'/L of finite separable extensions of k, the diagram

$$\mathscr{F}(L) \to \mathscr{F}(L') \rightrightarrows \mathscr{F}(L' \otimes_L L') \tag{3.7}$$

is an equaliser. If L'/L is Galois, it is easy to deduce the equality $\mathscr{F}(L) = \mathscr{F}(L')^{\mathrm{Gal}(L'/L)}$ from eq. (3.7), hence proving necessity.

Conversely, we first prove that $\mathscr{F}(L) = \mathscr{F}(L')^{\mathrm{Gal}(L'/L)}$ is equivalent to exactness of eq. (3.7) for Galois extensions.

($\Leftarrow$) We have natural maps

$$L' \underset{x \mapsto x \otimes 1}{\overset{x \mapsto 1 \otimes x}{\rightrightarrows}} L' \otimes_L L' \xrightarrow{\psi_\sigma} L' \tag{3.8}$$

where $\psi_\sigma \colon x \otimes y \mapsto x\sigma(y)$ for fixed $\sigma \in \mathrm{Gal}(L'/L)$. If $z \in \mathscr{F}(L')$ is in the equaliser of eq. (3.7), then $\mathscr{F}(\sigma)(z) = z$, as required.

($\Rightarrow$) If $z \in \mathscr{F}(L')^{\mathrm{Gal}(L'/L)}$, then ψ_σ is an isomorphism, so $\mathscr{F}(\psi_\sigma)$ is injective, which proves the exactness of eq. (3.7).

Finally, to show that exactness of eq. (3.7) for Galois extensions implies exactness for general extensions, consider the diagram

$$\begin{tikzcd}
\mathscr{F}(L) \arrow[r] \arrow[d, "\mathrm{Id}"] & \mathscr{F}(L') \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathscr{F}(L' \otimes_L L') \arrow[d] \\
\mathscr{F}(L) \arrow[r] & \mathscr{F}(L'') \arrow[r, shift left] \arrow[r, shift right] & \mathscr{F}(L'' \otimes_L L'')
\end{tikzcd} \tag{3.9}$$

where L''/L' is the Galois closure of an arbitrary finite separable extension L' over L. By assumption the bottom line is exact, and $\mathscr{F}(L) \to \mathscr{F}(L'')$ and $\mathscr{F}(L) \to \mathscr{F}(L')$ are easily seen be injective. A standard diagram chase then gives exactness of the top row. $\square$

We now turn to the construction of the functors of theorem 3.12. Suppose $\mathscr{F}$ is an étale sheaf and let $G := \mathrm{Gal}(k^{\mathrm{sep}}/k)$. Let $M_{\mathscr{F}} := \varinjlim \mathscr{F}(k')$ where k' runs over finite separable extensions of k. It is straightforward to check that the images of the inclusions $\mathscr{F}(k \hookrightarrow k')$ assemble to an injective system of abelian groups. Moreover, this is compatible with the action of G on each $\mathscr{F}(k')$, giving rise to an action of G on $M_{\mathscr{F}}$. Thus $M_{\mathscr{F}}$ is a G-module, and it is a good exercise to convince oneself that it is discrete.

Conversely, given $M \in \mathrm{Mod}(G)$, define a presheaf

$$\mathscr{F}_M\colon \mathrm{Alg}_{\text{ét}}(k) \to \mathrm{Ab} \quad \text{by} \quad \mathscr{F}_M(A) = \mathrm{Hom}_{\mathrm{Mod}(G)}(\mathscr{F}(A), M). \tag{3.10}$$

Here $\mathscr{F}(A) := \mathrm{Hom}_{\mathrm{Alg}(k)}(A, k^{\mathrm{sep}})$. By the fundamental theorem of Galois theory, for a finite separable extension k'/k we have $\mathscr{F}(k') \cong G/\mathrm{Gal}(k^{\mathrm{sep}}/k)$ as G-modules. It follows that $F_M(k') \cong M^{\mathrm{Gal}(k^{\mathrm{sep}}/k')}$. Note that $\mathscr{F}_M$ satisfies the criteria in lemma 3.13 for being an étale sheaf:

(i) $\mathscr{F}_M(\prod k_i) = \bigoplus \mathscr{F}_M(k_i)$ for finite collections of separable extensions k_i/k by the standard properties of Hom;
(ii) For k''/k' finite Galois, $\mathscr{F}_M(k'')^{\mathrm{Gal}(k''/k')} = \mathscr{F}_m(k')$ by the discussion above.

Exercise. Show that $M \mapsto F_M$ is fully faithful and essentially surjective.

This proves theorem 3.12. □

3.3 Henselian rings & étale stalks of the structure sheaf

Speaker: Jay Swar

When we first meet sheaves on topological spaces, a fundamental feature is that isomorphisms can be detected on the level of stalks. It is natural to ask whether the same holds for sheaves on sites, and in particular on the small étale site. First we need to extend the notion of points in a way which is compatible with our idea of coverings as distinguished X-schemes.

Definition 3.14. Let X be a scheme. A **geometric point** $\bar{x}$ is a map $\bar{x}\colon \operatorname{Spec}\Omega \to X$, where Ω is some separably closed field.

By definition, a geometric point $\bar{x}$ specifies a point $x \in X$ along with an embedding $\kappa(x) \hookrightarrow \Omega$. Note that for any étale covering U whose image contains x, the diagram

$$\begin{tikzcd} \operatorname{Spec}\Omega \arrow[r] \arrow[dr, "\bar{x}"'] & U \arrow[d] \\ & X \end{tikzcd} \tag{3.11}$$

commutes. Such a U is called an *étale neighbourhood*, and by abuse of notation we write $\overline{x} \in U$. We can now take the injective limit of sections over such étale neighbourhoods, giving the following definition:

Definition 3.15. The **étale stalk** of X at a geometric point $\overline{x}$ is given by

$$\mathcal{O}_{X,\overline{x}} := \varinjlim_{U \ni \overline{x}} \mathcal{O}_U(U). \tag{3.12}$$

Example 3.16. If $X = \operatorname{Spec} k$ for some field k and $\overline{x} : \operatorname{Spec} \Omega \to X$ is a geometric point, then $\mathcal{O}_{X,\overline{x}} \cong \Omega$.

The étale stalk satisfies many of the same properties as the usual stalk:

Proposition 3.17. *Let X be a scheme, $\overline{x}$ a geometric point and let $\kappa(\overline{x})$ denote the residue field of $\mathcal{O}_{X,\overline{x}}$.*

(i) *$\mathcal{O}_{X,\overline{x}}$ is a local ring.*
(ii) *$\mathcal{O}_{X,\overline{x}}$ is Noetherian.*
(iii) $\dim \mathcal{O}_{X,\overline{x}} = \dim \mathcal{O}_{X,x}$, *that is, the Krull dimension of the étale stalk is the same as that of the usual stalk.*
(iv) *every monic coprime factorisation in $\kappa(\overline{x})$ lifts to a factorisation in $\mathcal{O}_{X,\overline{x}}$.*
(v) *$\kappa(\overline{x})$ is separably closed.*

The last two properties do not hold for Zariski stalks, but are useful features of étale stalks.

Definition 3.18. A local ring which satisfies (iv) is said to be **Henselian**, and is **strictly Henselian** if it additionally obeys (v).

The Zariski stalks are not Henselian in general, so this is really a feature of the étale stalks. This is ample motivation to study Henselian local rings in general.

Proposition 3.19. *A local ring A with maximal ideal $\mathfrak{m}$ and residue field κ is Henselian if and only if for all $f_1, \dots, f_n \in R[x_1 \dots, x_n]$ with $\det\left(\frac{\partial \overline{f}_i}{\partial x_j}\right) \neq 0$, every common root of the reductions $\overline{f}_1, \dots, \overline{f}_n \in \kappa[x_1, \dots, x_n]$ lifts to a common root in R.*

This is reminiscent of Newton's method, or Hensel's lemma from which Henselian rings get their name.

Given any local ring, there is a canonical way to construct an associated Henselian ring, its *Henselisation*.

Proposition 3.20 ([Sta21, Lemma 04GN]). *Let A be a local ring. There exists a Henselian local ring A^h and $A \to A^h$ a local homomorphism such that for any local* *homomorphism $\phi\colon A \to B$ where B is a Henselian local ring, there exists a unique local homomorphism $A^h \to B$ such that the diagram commutes:*

$$\begin{tikzcd} A^h \arrow[dr, "\exists!"] & \\ A \arrow[u] \arrow[r] & B \end{tikzcd} \tag{3.13}$$

Recognising this as a universal property, the usual argument shows that A^h is unique up to unique isomorphism.

Proof. The idea here is to define a category consisting of pairs $(S, \mathfrak{q})$, where S is a ring equipped with an étale ring map $S \to A$, and $\mathfrak{q} \subset S$ is a prime lying above $\mathfrak{m}$, the maximal ideal of A. Morphisms in this category are given by A-algebra morphisms $\phi : S \to S'$ such that $\phi^{-1}(\mathfrak{q}') = \mathfrak{q}$. Then we can set $A^h := \varinjlim_{(S,\mathfrak{q})} S$, which exists because colimits exist in the category of rings. One also checks that this is Henselian; details are provided in the link above. □

Definition 3.21. The ring A^h is called the **Henselisation** of A.

With suitable modification to the argument in proposition 3.20, namely by considering instead triples $(S, \mathfrak{q}, \alpha)$ where α is a fixed map $\kappa(\mathfrak{q}) \to \kappa^{\mathrm{sep}}$, we get a strictly Henselian ring A^{sh}, called the **strict Henselisation** of A.

Example 3.22 (Exercise)**.** If $A = \mathbb{Z}_{(p)}$, then the strict Henselisation of A equals the integral closure of $\mathbb{Z}_{(p)}$ in $\mathbb{Z}_p$.

Example 3.23 (Exercise)**.** Let k be algebraically closed, and $A = \mathcal{O}_{\mathbb{A}_k,x}$ where x is the origin, corresponding to the prime ideal $(x_1, \dots, x_n)$. Then $A^h = k[[x_1, \dots, x_n]] \cap k(x_1, \dots, x_n)^{\mathrm{alg}}$.

Exercise ("You should probably google this"). Let X be a variety over an algebraically closed field k, $P \in X$ a non-singular point, and U some Zariski open neighbourhood of P. Then there exists an étale map $\phi\colon U \to \mathbb{A}_k^n$ sending P to 0.

Lemma 3.24. *Let k be algebraically closed, and let X and Y be k-varieties. If $X \to Y$ is an étale map, then the induced map $\mathcal{O}_{Y,f(\overline{x})} \to \mathcal{O}_{X,\overline{x}}$ is an isomorphism.*

Combining this with the previous, we get the following:

Corollary 3.25. *Étale stalks at non-singular points of k-varieties are isomorphic to the ring in example 3.23.*

This is explained by the following proposition:

Proposition 3.26. *Let X be a scheme, and $\overline{x}$ a geometric point of X with underlying point $x \in X$. Then $\mathcal{O}_{X,\overline{x}} \cong (\mathcal{O}_{X,x})^{sh}$.*

Example 3.27 ("Henselisation does not commute with fibre products"). Let L be a field of characteristic 0. Then $\mathcal{O}_{X,\overline{x}} = \overline{L}$, which contains more arithmetic information than just $L \cong \mathcal{O}_{X,x}$. On the other hand, if L/k is Galois, then

$$(\mathcal{O}_{X,\overline{x}})^h \otimes_k \overline{k} = \prod_i \overline{k}, \tag{3.14}$$

so in particular is not the strict Henselisation of k.

Theorem 3.28 (Artin approximation). *Let $\{f_i(x_1,\dots,x_n,y_1,\dots,y_n)\}_i \subset k[x_1,\dots,x_n,y_1,\dots,y_n]$ be a collection of polynomials, and let $\hat{y}_i$ be power series in x_i, i.e. $\hat{y}_i \in k[[[]]x_1,\dots,x_n]$, for $i = 1,\dots,n$. If $f_i(x_1,\dots,x_n,\hat{y}_1,\dots,\hat{y}_n) = 0$ for all i, then there is a collection of polynomials $y_1,\dots,y_n \in k[x_1,\dots,x_n]$ such that*

$$f(x_1,\dots,x_n,y_1,\dots,y_n) = 0 \quad \textit{and} \quad y_i \equiv \hat{y}_i \pmod{(x_1,\dots,x_n)^n}, \tag{3.15}$$

for all $i = 1,\dots,n$.

3.4 Stalks of étale sheaves

Speaker: Martin Gallauer

Let X be a locally Noetherian scheme, $\mathcal{F}$ a Zariski sheaf on X, and fix a point $x \in X$. The usual stalk of $\mathcal{F}$ at x can be described as the colimit $\varinjlim_U \mathcal{F}(U)$, where U runs over Zariski covers $U \xrightarrow{\iota} X$ such that

$$\begin{tikzcd} & U \arrow[d, "\iota"] \\ \operatorname{Spec}\kappa(x) \arrow[ur] \arrow[r, "\overline{x}"] & X \end{tikzcd} \tag{3.16}$$

commutes. If we identify U with an open subset of X, then this reduces to the requirement that $x \in U$. However, eq. (3.16) is much more amenable to generalisation in the relative setting.

Definition 3.29. Let X be a scheme, $\mathcal{F}$ a presheaf on $X_{\text{ét}}$, and $\overline{x} \colon \operatorname{Spec}\kappa(\overline{x}) \to X$ a geometric point of X. The **stalk of $\mathcal{F}$ at $\overline{x}$** is the object

$$\mathcal{F}_{\overline{x}} := \varinjlim_{(U,\overline{u})} \mathcal{F}(U), \tag{3.17}$$

where U ranges over étale schemes $U \to X$ along with geometric points $\overline{u}$ of U such that the associated diagrams

$$\begin{tikzcd} & U \arrow[d] \\ \operatorname{Spec}\kappa(x) \arrow[ur, "\overline{u}"] \arrow[r, "\overline{x}"] & X \end{tikzcd} \tag{3.18}$$

commute.

A pair $(U,\overline{u})$ is frequently referred to as an *étale neighbourhood of* $\overline{x}$.

Remark. (i) If $\mathscr{F}$ is a presheaf valued in C, where C is the category of abelian groups, rings or modules, then $\mathscr{F}_{\overline{x}}$ is an object of C; this is equivalent to n the statement that $(\cdot)_{\overline{x}}\colon \mathscr{F} \mapsto \mathscr{F}_{\overline{x}}$ is a map $\mathrm{Sh}(X_{\text{ét}}) \to \mathrm{C}$. In fact, (exercise!) it naturally determines a functor.

(ii) In the situation above, the colimit eq. (3.17) is *filtered*[12], and it follows (exercise!) that $(\cdot)_{\overline{x}}$ is an exact functor.

(iii) The stalk $\mathscr{F}_{\overline{x}}$ only depends on the choice of separable closure $\kappa(\overline{x})$ up to isomorphism.

Proposition 3.30. *A sequence of étale sheaves $\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ is exact if and only if for every geometric point $\overline{x}$ of X, the associated sequence $\mathscr{F}_{\overline{x}} \to \mathscr{G}_{\overline{x}} \to \mathscr{H}_{\overline{x}}$ is exact.*

Proof sketch. This is mostly a formal verification. The key point is to reduce to the following statement: If $U \to X$ is étale, $\mathscr{P} \in \mathrm{Sh}(X_{\text{ét}})$, $s \in \mathscr{P}(U)$ and $s_{\overline{x}} = 0$ for all geometric points $\overline{x}$ of X, then $s = 0$. Let's prove this:

Since $s_{\overline{x}} = 0$, by definition there exists some étale neighbourhood $V_{\overline{u}} \to U$ of $\overline{u}$ such that $s|_{V_{\overline{u}}} = 0$.

$$\begin{tikzcd} & V_{\overline{u}} \arrow[d] \\ & U \arrow[d] \\ \operatorname{Spec}\kappa(\overline{x}) \arrow[r] \arrow[ur, "\overline{u}"] \arrow[uur, bend left, "\overline{v}"] & X \end{tikzcd} \tag{3.19}$$

But then the collection $(V_{\overline{u}} \to U)_{\overline{u}}$ is an étale covering, and so $s = 0$ by the sheaf condition. $\square$

Recall that if $\mathscr{F} = \mathscr{O}_X$ and $\overline{x}$ is any geometric point of X with image x, then $\mathscr{O}_{X,\overline{x}} = (\mathscr{O}_{X,x})^{sh}$. These fit into a diagram

$$\begin{tikzcd} \mathscr{O}_{X,x} \arrow[r] \arrow[dr] & \mathscr{O}^h_{X,x} \arrow[r] \arrow[d] & \mathscr{O}^{sh}_{X,x} \cong \mathscr{O}_{X,\overline{x}} \\ & \widehat{\mathscr{O}}_{X,x} & \end{tikzcd} \tag{3.20}$$

Another important feature of stalks in the étale topology is that they admit a natural Galois action. More precisely, if $\kappa(\overline{x})/\kappa(x)$ is the separably closed field extension associated to a geometric point $\overline{x}$ and $\mathscr{F}$ an étale sheaf, then $G := \mathrm{Gal}(\kappa(\overline{x})/\kappa(x))$ acts on $\mathscr{F}_{\overline{x}}$ as follows: for any $\sigma \in G$, a triple $(U,\overline{u},s)$ where $(U,\overline{u})$ is an étale neighbourhood of $\overline{x}$ and $s \in \mathscr{F}(U)$ is sent to the triple $(U,\overline{u}\circ\sigma,s)$.

[12] i.e. the colimit over a filtered category, see here.

Exercise. Check that this induces an action on the stalk $\mathcal{F}_{\bar{x}} = \varinjlim_{(U,\bar{u})} \mathcal{F}(U)$.

As a consequence, the functor $(\cdot)_{\bar{x}}$ is actually a functor $\mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Mod}(G)$, the category of G-modules; it is easy to verify that morphisms are automatically G-equivariant.

Exercise. Prove that this is in fact an equivalence of categories, by showing that $(\cdot)_{\bar{x}}$ coincides with the functor of theorem 3.12.

3.5 Operations on sheaves

Speaker: Eduardo de Lorenzo Poza

In this section we prove that many of the operations on sheaves generalises from the usual setting. First we need to extend our notion of a continuous map:

Definition 3.31. Let $(\mathrm{C}'/X')_{E'}$ and $(\mathrm{C}/X)_E$ be sites of schemes X and X', and $\pi\colon X' \to X$ a morphism of schemes. We say that π is a **continuous map of sites** if the following conditions hold:

(i) If $Y \in \mathrm{C}$, then $Y_{(X')} := Y \times_X X' \in \mathrm{C}'$.
(ii) In the Cartesian diagram

$$\begin{tikzcd} U_{(X')} \arrow[r] \arrow[d, "f'"] & U \arrow[d, "f"] \\ X' \arrow[r, "\pi"] & X \end{tikzcd} \tag{3.21}$$

if $f \in E$, then the base change f' is in E'.

Here (i) is an analogue of the property that preimages of open sets are open, and by abuse of notation we write $\pi^{-1}(Y) := Y_{(X')} = Y \times_X X'$. On the other hand, (ii) ensures that we don't run into trouble when pulling back covers.

Note that since base change preserves surjectivity (see eg. [Sta21, Lemma 01S1]) a continuous map of sites takes coverings to coverings.

Example 3.32. Any morphism $X' \to X$ induces a continuous map of sites $X'_{\text{ét}} \to X_{\text{ét}}$; this is a direct consequence of proposition 2.44.

Definition 3.33. Let $\pi\colon X'_{E'} \to X_E$ be a continuous map of sites, and $\mathcal{F}'$ a presheaf on $X'_{E'}$. The **direct image presheaf** $\pi_p\mathcal{F}'$ is the presheaf on X_E defined by $\pi_p\mathcal{F}'(U) := \mathcal{F}'(U \times_X X')$.

Note that if $\mathcal{F}'$ is a sheaf, then so is $\pi_p\mathcal{F}'$. In fact, the map π_p is a functor $\mathrm{pSh}(X'_{E'}) \to \mathrm{pSh}(X_E)$. While it is not hard to check that π_p preserves exactness, it is not true for the restriction to the full subcategory of sheaves, $\pi_*\colon \mathrm{Sh}(X'_{E'}) \to \mathrm{Sh}(X_E)$.

As in the case of sheaves on a topological space we can also pull a sheaf on X back along $X' \to X$:

Definition 3.34. Let $\pi : X'_{E'} \to X_E$ be a continuous map of sites. The **inverse image functor** is the functor $\pi^p\colon \mathrm{pSh}(X_E) \to \mathrm{pSh}(X'_{E'})$ given by the left adjoint of π_p.

The existence of such a functor follows from a general category-theoretical argument, and more details can be found in [Mil80, II.2.2].

In the étale topology, we can give a more explicit construction of π^p: For $\mathcal{F} \in \mathrm{pSh}(X_{\text{ét}})$, let $\pi^p\mathcal{F}(U') := \varinjlim_{(g,U)} \mathcal{F}(U)$, where the colimit is taken over pairs (g, U) fitting into a commuting diagram

$$\begin{tikzcd} U' \arrow[r, "g"] \arrow[d] & U \arrow[d] \\ X' \arrow[r, "\pi"] & X \end{tikzcd} \tag{3.22}$$

These form a direct system with morphisms $h\colon U_1 \to U_2$ fitting into the commutative diagrams

$$\begin{tikzcd} U' \arrow[rr, "g_2"] \arrow[dr, "g_1"] \arrow[dd] & & U_2 \arrow[dd] \\ & U_1 \arrow[ur, "h"] \arrow[dr] & \\ X' \arrow[rr, "\pi"] & & X \end{tikzcd} \tag{3.23}$$

One can check by hand that this indeed defines a sheaf on $X'_{\text{ét}}$, and that it is left adjoint to π_p. Note that for general sites X_E and $X'_{E'}$ the functor π^p does not preserve the sheaf condition. See [Mil80, §II.2] for further details.

Proposition 3.35. *Let $\pi\colon X'_{E'} \to X_E$ be a continuous map of sites.*

(i) *The functor $\pi_p\colon \mathrm{pSh}(X'_{E'}) \to \mathrm{pSh}(X_E)$ is exact;*
(ii) *the functor $\pi^p\colon \mathrm{pSh}(X_E) \to \mathrm{pSh}(X'_{E'})$ is right exact;*
(iii) *the functor $\pi^p\colon \mathrm{Sh}(X_E) \to \mathrm{pSh}(X'_{E'})$ is also left exact in the étale topology.*

Proof. (i) follows directly from definition; simply check every étale $U \to X$. (ii) follows from adjointness. Finally, (iii) follows from the fact that $\pi^p\mathcal{F}(U')$ is a *cofiltered colimit* in Ab, and these are exact by a general category-theoretic argument. ∎

Example 3.36 (The Kummer sequence)**.** Recall that the étale sheaf μ_n from example 3.7 is represented by

$$\operatorname{Spec} \frac{\mathbb{Z}[t, t^{-1}]}{(t^n - 1)} \times_{\operatorname{Spec}\mathbb{Z}} X. \tag{3.24}$$

This can equivalently be realised as the kernel sheaf of the map $\mathbb{G}_m \to \mathbb{G}_m$, where $\mathbb{G}_m$ is the multiplicative group scheme represented by $\mathbb{G}_{m,X} := \operatorname{Spec}\mathbb{Z}[t, t^{-1}] \times X$; explicitly we have $\mathbb{G}_m(U) = \Gamma(\mathcal{O}_U, U)^\times$. The corresponding sequence

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{s \mapsto s^n} \mathbb{G}_m \to 0 \tag{3.25}$$

need not be exact in general. This is called the *Kummer sequence*.

Exercise. Suppose n is invertible everywhere on X.

(i) Show that the Kummer sequence is not exact on Zariski sheaves.
(ii) Show that the Kummer sequence is not exact on Zariski presheaves.
(iii) Show that the Kummer sequence is exact in the category of étale sheaves.

Example 3.37 (The Artin-Schreier sequence)**.** Let X be a scheme over a field of characteristic p, and let $\mathbb{G}_a$ be the sheaf on $X_{\text{ét}}$ given by $\mathbb{G}_a(U) = \Gamma(\mathcal{O}_U, U)$. Then we have a sequence

$$0 \to \underline{\mathbb{Z}/p\mathbb{Z}} \to \mathbb{G}_a \xrightarrow{F - \mathrm{Id}} \mathbb{G}_a \to 0, \tag{3.26}$$

where F is the Frobenius map on $\mathbb{G}_a$. This is called the *Artin-Schreier sequence*. As with the Kummer sequence, this is not exact on the right in the categories of Zariski sheaves or presheaves, but it is exact in the category of étale sheaves, essentially because the polynomial $T^p - T$ is separable (exercise!).

Just like with topological spaces, there is a canonical way of producing a sheaf on any site from a given presheaf.

Theorem 3.38. *Let X_E be a site. For any presheaf $\mathcal{P} \in \mathrm{pSh}(X_E)$, there exists a sheaf $\mathcal{P}^a \in \mathrm{Sh}(X_E)$ such that for any sheaf $\mathcal{F} \in \mathrm{Sh}(X_E)$ and morphism of presheaves $\mathcal{P} \to \mathcal{F}$, there exists a unique morphism of sheaves $\mathcal{P}^a \to \mathcal{F}$ so that the following diagram commutes:*

$$\begin{tikzcd} & \mathcal{P}^a \arrow[d, dashed, "\exists!"] \\ \mathcal{P} \arrow[ur, "a"] \arrow[r] & \mathcal{F} \end{tikzcd} \tag{3.27}$$

For a proof, see [Mil80, Thm. II.2.11] or [Sta21, Section 00W1].

Since a is defined by a universal property, the standard argument shows that $\mathcal{F}^a$ is unique up to unique isomorphism. In fact, the construction is functorial.

Definition 3.39. The functor $\mathcal{F} \mapsto \mathcal{F}^a$ is called the **sheafification functor**.

One important fact about sheafification is that it preserves stalks: for any geometric point $\bar{x}$ on X, we have that $(\mathscr{F}^a)_{\bar{x}} = \mathscr{F}_{\bar{x}}$.

Proposition 3.40. *Let X_E be a site.*

(i) *Sheafification is functorial, and the sheafification functor a is left adjoint to the inclusion functor $\mathrm{Sh}(X_E) \hookrightarrow \mathrm{pSh}(X_E)$. Moreover, the functor a is exact.*
(ii) *For a sequence of sheaves $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$, exactness on the left in $\mathrm{Sh}(X_E)$ is equivalent to left exactness as a sequence of presheaves. This is also equivalent to left exactness on the level of sections; if $X_E = X_{\text{ét}}$, then this is also equivalent to left exactness on stalks.*
(iii) *A morphism in $\mathrm{Sh}(X_E)$ is surjective if and only if it is surjective on sections. If $X_E = X_{\text{ét}}$, then this is equivalent to surjectivity on stalks.*
(iv) *Limits in $\mathrm{Sh}(X_E)$ coincide with limits in $\mathrm{pSh}(X_E)$; a colimit in $\mathrm{Sh}(X_E)$ is the sheafification of a colimit in $\mathrm{pSh}(X_E)$*
(v) *The category $\mathrm{Sh}(X_E)$ is abelian and has arbitrary direct sums and products; filtered colimits respect exactness.*

With the axioms from Grothendieck's Tohoku paper, (v) can be rephrased as "$\mathrm{Sh}(X_E)$ satisfies AB5, AB3* (but not AB4*)".[13]

Let $\pi\colon X' \to X$ be a continuous map of sites. Recall that we defined the functor π_* as the restriction of π_p to the category of sheaves. This fails for π^p because $\pi^p\mathscr{F}$ is generally not a sheaf even if $\mathscr{F}$ is. However, we can mend this by sheafifying:

Definition 3.41. Let $\pi\colon X' \to X$ be a continuous map of sites. The functor

$$\pi^*\colon \mathrm{Sh}(X_E) \to \mathrm{Sh}(X'_{E'}), \qquad \mathscr{F} \mapsto \pi^*\mathscr{F} := (\pi^p\mathscr{F})^a \tag{3.28}$$

is called the **pullback along** π, or **inverse image functor**.

By the universal property of sheafification, it is easy to see that (π_*, π^*) form an adjoint pair. Since π^* is not exact in general, neither is π^*. However, by proposition 3.35 π^p *is* for the étale site, and so π^* is as well, being a composition of left exact functors, $\mathrm{Sh} \hookrightarrow \mathrm{pSh} \xrightarrow{\pi^p} \mathrm{pSh} \xrightarrow{a} \mathrm{Sh}$.

For the remainder of the section, we fix a scheme X equipped with the étale topology. Moreover, if $j\colon U \hookrightarrow X$ is an open immersion, we tend to identify U with its image in X. The following proposition tells us how pullbacks and pushforwards interact with stalks.

Proposition 3.42 ([Mil80, Cor. II.3.5])**.** *Let X and X' be schemes.*

[13] AB4* states that an arbitrary product of exact sequences is exact, which is false in general. For some counterexamples, see here.

(i) *For any* $\pi \colon X' \to X$, $\mathscr{F} \in \mathrm{Sh}(X_{\text{ét}})$, *and* $\overline{x}'$ *a geometric point on* X'. *Then* $(\pi^*\mathscr{F})_{\overline{x}'} = \mathscr{F}_{\overline{\pi(x')}}$.

(ii) *If* $j \colon U \to X$ *is an open immersion,* $\mathscr{F} \in \mathrm{Sh}(U_{\text{ét}})$, *and* $\overline{x}$ *a geometric point on* X *such that* $x \in U$, *then* $(j_*\mathscr{F})_{\overline{x}} = \mathscr{F}_{\overline{x}}$.

(iii) *If* $i \colon Z \to X$ *is a closed immersion,* $\mathscr{F} \in \mathrm{Sh}(Z_{\text{ét}})$, *and* $\overline{x}$ *a geometric point on* X, *then*

$$(i_*\mathscr{F})_{\overline{x}} = \begin{cases} \mathscr{F}_{\overline{x}} & \text{if } x \in Z, \\ 0 & \text{if } x \notin Z. \end{cases} \tag{3.29}$$

(iv) *If* $\pi \colon X' \to X$ *is finite and* $\mathscr{F}' \in \mathrm{Sh}(X')$, *then* $(\pi_*\mathscr{F}')_{\overline{x}} = \bigoplus_{x' \mapsto x} (\mathscr{F}'_{\overline{x}})^{d(x')}$, *where* $d(x') = [\kappa(x') : \kappa(\overline{x})]_{\text{sep}}$, *for any geometric point* $\overline{x}$ *of* X.

Definition 3.43. Let $j \colon U \hookrightarrow X$ be an open immersion of schemes, and fix $\mathscr{P} \in \mathrm{pSh}(U_{\text{ét}})$. Define

$$\mathscr{P}_!(V) := \begin{cases} \mathscr{P}(V) & \text{if } \phi(V) \subset U \text{ for an étale morphism } \phi \colon V \to X, \\ 0 & \text{otherwise.} \end{cases} \tag{3.30}$$

This is called "$\mathscr{P}$ **lower shriek**".

If $f \colon \mathscr{P} \to \mathscr{P}'$ is a morphism of presheaves, then we obtain an associated morphism $\mathscr{P}_! \to \mathscr{P}'_!$ by "extending f by 0 outside U"; thus $\mathscr{P} \mapsto \mathscr{P}_!$ is a functor. We can upgrade this to a functor of sheaves by precomposing with the inclusion $\mathrm{Sh} \hookrightarrow \mathrm{pSh}$ and postcomposing with sheafification.

Definition 3.44. Let $U \hookrightarrow X$ be an open immersion of schemes. The **extension by** 0**-functor** $j_! \colon \mathrm{Sh}(U_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$ is given by $\mathscr{F} \mapsto j_!\mathscr{F} := (\mathscr{F}_!)^a$.

It is a straightforward exercise using the universal property of sheafification to show that $(j_!, j^*)$ form an adjoint pair.

Proposition 3.45. *If* $j \colon U \hookrightarrow X$ *is an open immersion,* $\mathscr{F} \in \mathrm{Sh}(U_{\text{ét}})$ *and* $\overline{x}$ *a geometric point of* X, *then* $(j_!\mathscr{F})_{\overline{x}} = \mathscr{F}_{\overline{x}}$ *if* $x \in U$, *and* 0 *otherwise.*

Definition 3.46. Let $i \colon Z \to X$ be a closed immersion and $j \colon U = X \setminus Z \to X$ an open immersion. We define a category $T(X)$ consisting of triples $(\mathscr{F}_1, \mathscr{F}_2, \phi)$ where $\mathscr{F}_1 \in \mathrm{Sh}(Z_{\text{ét}}), \mathscr{F}_2 \in \mathrm{Sh}(U_{\text{ét}})$ and $\phi \colon \mathscr{F}_1 \to i^*j_*\mathscr{F}_2$ is a morphism in $\mathrm{Sh}(Z_{\text{ét}})$.

Theorem 3.47 ([Mil80, Thm. II.3.10])**.** *Fix* $i \colon Z \to X$ *a closed immersion and* $j \colon U = X \setminus Z \to X$ *an open immersion. There is an equivalence of categories*

$$\begin{aligned} \mathrm{Sh}(X_{\text{ét}}) &\to T(X) \\ \mathscr{F} &\mapsto (i^*\mathscr{F}, j^*\mathscr{F}, \phi_{\mathscr{F}}) \\ \psi &\mapsto (i^*\psi, j^*\psi). \end{aligned}$$

Definition 3.48. Let X be a scheme, $Y \hookrightarrow X$ a subscheme and $\mathscr{F} \in \mathrm{Sh}(X_{\text{ét}})$. We say that $\mathscr{F}$ **has support in** Y if $\mathscr{F}_{\overline{x}} = 0$ for every geometric point $\overline{x}$ with image in $X \setminus Y$.

From the previous theorem we deduce the following:

Corollary 3.49 ([Mil80, Cor. II.3.11])**.** *With notation as above, there is an equivalence of categories between* $\mathrm{Sh}(Z_{\text{ét}})$ *and the full subcategory of sheaves on* X *with support in* Z*.*

Proof (sketch). The main idea here is to show that sheaves with support in Z are equivalent to the subcategory of $T(X)$ given by $(i^*\mathscr{F}, 0, 0)$. □

Definition 3.50. Let $i\colon Z \to X$ be a closed immersion and $j\colon U = X \setminus Z \to X$ an open immersion. Then we have functors

$$\begin{tikzcd} \mathrm{Sh}(Z_{\text{ét}}) \arrow[r, "i_*"] & \mathrm{Sh}(X_{\text{ét}}) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathrm{Sh}(U_{\text{ét}}) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{3.31}$$

which using the equivalence in corollary 3.49 are given explicitly as follows:

$$\begin{aligned} i^* &: \mathscr{F}_1 \leftarrow (\mathscr{F}_1, \mathscr{F}_2, \phi), & j_! &: (0, \mathscr{F}_2, 0) \leftarrow \mathscr{F}_2, \\ i_* &: \mathscr{F}_1 \mapsto (\mathscr{F}_1, 0, 0), & j^* &: (\mathscr{F}_1, \mathscr{F}_2, \phi) \mapsto \mathscr{F}_2, \\ i^! &: \ker\phi \leftarrow (\mathscr{F}_1, \mathscr{F}_2, \phi), & j_* &: (i^* j_* \mathscr{F}_2, \mathscr{F}_2, \mathrm{Id}) \leftarrow \mathscr{F}_2. \end{aligned} \tag{3.32}$$

Proposition 3.51 ([Mil80, Prop. II.3.14])**.** *Keeping the notation from the previous definition, we have the following:*

(i) *For the top four functors in eq. (3.32), each forms an adjoint pair with the one immediately below.*
(ii) *The functors i^*, i_*, j^* and j_* are exact.*
(iii) *The functors i_*, j_* and $j_!$ are fully faithful.*

4 Cohomology

Contents

4.1 Cohomology on sites

Speaker: Lukas Kofler

We assume some familiarity with the cohomological machinery used in algebraic geometry, and only give a quick summary here to fix notations. Further details can be found in [Har77, Chap. III].

Let $\mathscr{A}$ be an abelian category, and recall that an object I of $\mathscr{A}$ is *injective* if the functor $\mathrm{Hom}(-,I)$ is exact. We say $\mathscr{A}$ *has enough injectives* if for every element of $\mathscr{A}$ there exists an injection $A \hookrightarrow I$ where I is injective.

Proposition 4.1 ([Mil80, Prop. III.1.1]). *The category of sheaves valued in abelian groups on a site has enough injectives.*

In any abelian category $\mathscr{A}$ along with a left exact functor $F\colon \mathscr{A} \to \mathscr{A}'$, we can form right derived functors R^iF, $i \geq 0$, in the usual manner. These are characterised by the properties $R^0F = F$, $R^iF(I) = 0$ for any injective object I, and that every short sequence in $\mathscr{A}$ gives a long exact sequence in cohomology in $\mathscr{A}'$.

Example 4.2. The *global sections functor* $\Gamma(X,-)\colon \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Ab}$ is left exact, and we define $H^i(X,-) := R^i\Gamma(X,-)$ to be the corresponding cohomology functors.

Example 4.3. The inclusion $\mathrm{Sh}(X_{\text{ét}}) \hookrightarrow \mathrm{pSh}(X_{\text{ét}})$ is left exact by proposition 3.40 (i), and the cohomology functors are denoted by $\underline{H}^i(-)$.

Example 4.4. For a fixed sheaf $\mathscr{F} \in \mathrm{Sh}(X_{\text{ét}})$, the functor $\mathrm{Hom}(-,\mathscr{F})$ is left exact, and the right derived functors are denoted by $\mathrm{Ext}^i_{\mathrm{Sh}(X_{\text{ét}})}(\mathscr{F},-)$.

Example 4.5. Similarly, for $\mathscr{F},\mathscr{G} \in \mathrm{Sh}(X_{\text{ét}})$ we can define the hom-sheaf by

$$\mathscr{H}om(\mathscr{F},\mathscr{G})\colon U \mapsto \mathrm{Hom}(\mathscr{F}|_U,\mathscr{G}|_U). \tag{4.1}$$

This gives a left exact functor $\mathscr{H}om(\mathscr{F},-)\colon \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$, with right derived functors $\mathscr{E}xt^i(\mathscr{F},-)$.

Example 4.6. For a continuous map of sites $\pi\colon X'_{E'} \to X_E$, the pushforward π_* is left exact, and the right derived functors $R^i\pi_*$ are called *higher direct images*.

4.2 Spectral sequences

Spectral sequences have a reputation for being somewhat arcane objects, and so we begin the section gently with some motivation:

Suppose we have a double complex $\{E_0^{p,q}\}_{p,q\geq 0}$ in an abelian category $\mathscr{A}$; that is, a collection of objects $E_0^{0,0}, E_0^{1,0}, E_0^{0,1}, \ldots$ along with maps

$$d_h\colon E_0^{p,q} \to E_0^{p+1,q} \quad \text{and} \quad d_v\colon E_0^{p,q} \to E_0^{p,q+1} \tag{4.2}$$

satisfying $d_h^2 = 0 = d_v^2$ and $d_h d_v = -d_v d_h$. These arise naturally in algebraic geometry, say from taking resolutions of complex, or complexes of filtered objects. From this double complex we construct the *total complex* $E_0^\bullet$ with $E_0^k := \bigoplus_i E^{i,k-i}$, the direct sum along the k-th antidiagonal. This is becomes a complex with the differential $d := d_h + d_v$.

It is natural to ask whether one can find the cohomology of the total complex by computing cohomology of the complexes in the horisontal or vertical directions separately. Taking cohomology of $E_0^{\bullet,\bullet}$ first in the vertical direction under the action of d_v, we define

$$E_1^{p,q} := \frac{\ker d_v^{p,q}}{\operatorname{im} d_v^{p,q-1}}. \tag{4.3}$$

This gives a new double complex, where the action of the induced maps d_v is trivial. However, the induced maps $d_h : E_1^{p,q} \to E_1^{p+1,q}$ are well-defined (check!) and non-zero in general. By convention they are denoted d_1, and $E_1^{\bullet,\bullet}$ is called *the first page*. We can now take its cohomology under d_1, and the resulting double complex is denoted by $E_2^{\bullet,\bullet}$, called *the second page*.

Now, one might think that we are done at this point, and should be able to say something about the cohomology of the total complex. In fact, if the only non-zero columns of $E_2^{\bullet,\bullet}$ are given by p and $p+1$, then we have an exact sequence

$$0 \to E_2^{p,q} \to H^{p+q}(E^\bullet) \to E_2^{p+1,q-1} \to 0, \tag{4.4}$$

so we have computed $H^n(E^\bullet)$ "up to extension".

However, in general there is a new non-zero differential on $E_2^{\bullet,\bullet}$, $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$ constructed by the following diagram chase:

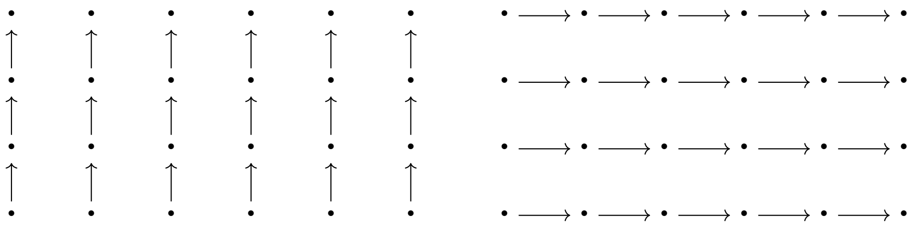
Take $x \in E_2^{p,q}$ and lift it to $x' \in E_1^{p,q}$. Then $d_1(x') = 0$ in $E_1^{p+1,q}$, so for a lift $x'' \in E_0^{p,q}$ of x', $d_h(x'')$ is in the image of d_v, say $d_h(x'') = d_v(y)$ for $y \in E_0^{p+1,q-1}$. Now $d_h(y) \in E_0^{p+2,q-1}$ and $d_v d_h(y) = -d_h d_v(y) = -d_h^2(x'') = 0$, so $d_h(y) \in \ker d_v$, determining an element of $E_1^{p+2,q-1}$. Since $d_1 d_h)(y) = 0$, this factors through to an element of $E_2^{p+2,q-1}$, which is the desired image of x.

Now that we have defined the map, it is not too difficult to check that it is well-defined and a differential, and in fact this construction generalises to higher differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$. This leads to the following definition:

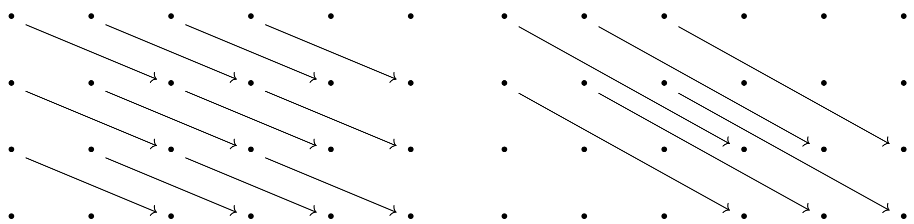
Definition 4.7. A (cohomological, first quadrant) **spectral sequence** consists of

(i) objects $E_r^{p,q} \in \mathcal{A}$ for all $p, q, r \geq 0$,
(ii) morphisms $d_r \equiv d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ satisfying $d_r^2 = 0$,
(iii) isomorphisms $\ker d_r^{p,q} / \operatorname{im} d_r^{p-r,q-r+1} \cong E_{r+1}^{p,q}$.

The collection $\{E_r^{p,q}\}_{p,q\geq 0}$ is called **the r-th page** of the spectral sequence.



E_0 and E_1.



E_2 and E_3.

Note that since $E_r^{p,q}$ all lie in the upper quadrant, we eventually take the cohomology of $0 \to E_r^{p,q} \to 0$, the zeroes lying in the second and fourth quadrant respectively. When this happens, we evidently have $E_r^{p,q} = E_{r+1}^{p,q} = \ldots$, and we write $E_r^{p,q} = E_\infty^{p,q}$.

Theorem 4.8. *For each $n \geq 0$, there is a decreasing filtration on $H^n(E^\bullet)$,*

$$H^n = F^0 H^n \supset \ldots \supset F^{n+1} H^n = 0, \tag{4.5}$$

such that $\mathrm{gr}_p H^n = E_\infty^{p,n-p}$.[14]

In particular, we have that $\bigoplus_{p=0}^n E_\infty^{p,n-p} = \mathrm{gr}\, H^n$. We write $E_0^{p,q} \Rightarrow H^{p+q}(E^\bullet)$ and say that the spectral sequence **converges to** $H^{p+q}(E^\bullet)$.

Note that we have not quite computed the cohomology of the total complex, but if for some $r \geq 2$ we have that $E_r^{\bullet,\bullet}$ has only one non-zero column or row, then we can read off $H^n(E^\bullet)$ directly. In this case we say that the spectral sequence **collapses, or degenerates, at page** r. In most applications, spectral sequences already collapse at E_1 or E_2.

A powerful feature of spectral sequences is that we can flip the roles of d_h and d_v while still converging to the cohomology of the graded complex. For convenience, let $\hat{E}$ denote the original spectral sequence and $\vec{E}$ the one with d_h and d_v swapped, and let's look at some applications:

[14] Recall that the *p-th graded part* of the filtered object $F^\bullet H^n$ is given by the quotient $F^{p+1}H^n / F^p H^n$, which is the p-th summand of $\mathrm{gr}\, H^n := \bigoplus_{k \geq 0} F^{k+1} H^n / F^k H^n$.

Example 4.9 (Five lemma)**.** Suppose we have the following diagram

$$\begin{tikzcd}
F \arrow[r] & G \arrow[r] & H \arrow[r] & I \arrow[r] & J \\
A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & D \arrow[r] \arrow[u, "\delta"] & E \arrow[u, "\epsilon"]
\end{tikzcd} \tag{4.6}$$

where the rows are exact, and α, β, δ and ϵ are isomorphisms. The *five lemma* states that in this case γ is an isomorphism as well. We can show this using a spectral sequence argument: view the diagram in eq. (4.6) as $\vec{E}_0^{p,q}$, and take horisontal cohomology to get $\vec{E}_1^{p,q}$, which since the rows are exact looks as follows:

$$\begin{tikzcd}
? & 0 & 0 & 0 & ? \\
? \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & ? \arrow[u]
\end{tikzcd} \tag{4.7}$$

Now the cohomology of the total complex vanishes in the degrees corresponding to $H \to C$. The spectral sequence converges at the 2nd page since there are no more arrows between non-zero objects to draw there.

Now let's look at the vertical cohomology. The first page, $\hat{E}_1^{p,q}$, looks as follows:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd} \tag{4.8}$$

and γ being an isomorphism is equivalent to the vanishing of the two question marks here. Note that the spectral sequence converges on this page, and so since the question marks correspond to the same pieces of the cohomology which vanished by the previous computation, we conclude that γ is indeed an isomorphism. This proves the claim.

Example 4.10 (Long exact sequence in cohomology)**.** Using spectral sequences we can also deduce the long exact sequence in cohomology from a short exact sequence of objects. Suppose

$$0 \to A \to B \to C \to 0 \tag{4.9}$$

is an exact sequence. In horisontal cohomology the sequence converges on the first page because the sequence is exact. On the other hand, $\hat{E}_1^{p,q}$ is given by

$$\begin{tikzcd}
0 \arrow[r] & H^2(A) \arrow[r, "\alpha_2"] & H^2(B) \arrow[r, "\beta_2"] & H^2(C) \arrow[r] & 0 \\
0 \arrow[r] & H^1(A) \arrow[r, "\alpha_1"] & H^1(B) \arrow[r, "\beta_1"] & H^1(C) \arrow[r] & 0 \\
0 \arrow[r] & H^0(A) \arrow[r, "\alpha_0"] & H^0(B) \arrow[r, "\beta_0"] & H^0(C) \arrow[r] & 0
\end{tikzcd} \tag{4.10}$$

and the next page looks like this:

$$
\begin{tikzcd}
0 & \ker\alpha_2 \arrow[drr] & \ker\beta_2/\operatorname{im}\alpha_2 & \operatorname{coker}\beta_2 & 0 \\
0 & \ker\alpha_1 \arrow[drr] & \ker\beta_1/\operatorname{im}\alpha_1 & \operatorname{coker}\beta_1 & 0 \\
0 & \ker\alpha_0 & \ker\beta_0/\operatorname{im}\alpha_0 & \operatorname{coker}\beta_0 & 0
\end{tikzcd}
\tag{4.11}
$$

The sequence converges on the next page, and so we conclude that every entry on the next page is 0. In particular, $\ker\beta_i/\operatorname{im}\alpha_i = 0$ and $\ker\alpha_{i+1} \cong \operatorname{coker}\beta_i$ for all $i \geq 0$. This gives the connecting homomorphism $H^i(C) \to H^{i+1}(A)$ along with exactness everywhere in the long exact sequence.

Exercise. Prove the snake lemma using spectral sequences: given a commutative diagram with exact rows

$$
\begin{tikzcd}
0 \arrow[r] & E \arrow[r] & F \arrow[r] & G \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}
\tag{4.12}
$$

prove the exactness of the sequence

$$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \to \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma \to 0. \tag{4.13}$$

Next we turn to study properties of spectral sequences. For convenience, we consider only sequences $E = \hat{E}$, with d_0 vertical. We will also abstract slightly and let $E^{\bullet,\bullet}$ converge to any family of filtered objects $E^n \in \mathcal{A}$ such that $F^0E^n = E^n$ and $F^{n+1}E^n = 0$. As usual we then have $E_\infty^{p,q} \cong \operatorname{gr}_p E^{p+q}$.

Note that $E_{r+1}^{p,q}$ is a subquotient of $E_r^{p,q}$ for all p, q, r; this gives rise to a sequence of quotient maps

$$E_0^{n,0} \to E_1^{n,0} \to \dots \to E_\infty^{n,0}. \tag{4.14}$$

The natural composite $E_0^{n,0} \to E^n$ is called an *edge morphism*. In a similar manner we construct an edge morphism $E^n \to E_0^{0,n}$.

Exercise. Show that the following sequence is exact:

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \xrightarrow{d} E_2^{2,0} \to E^2. \tag{4.15}$$

This is called the **five term exact sequence**.

Example 4.11. The Hochschild-Serre spectral sequence in group cohomology computes the group cohomology of a group G in terms of a subgroup H and the quotient G/H. In this case, the five term exact sequence is simply the inflation-restriction sequence

$$0 \to H^1(G/H, A^H) \to H^1(G,A) \to H^1(H,A)^{G/H} \to H^2(G/H, A^H) \to H^2(G,A). \tag{4.16}$$

We round off the section with a theorem, the "chain rule for derived functors", which we will put to good use later:

Theorem 4.12 (The Grothendieck spectral sequence)**.** *Let* A, B *and* C *be abelian categories, with* A *and* B *having enough injectives. Suppose we are given left exact functors* $\mathrm{A} \xrightarrow{G} \mathrm{B} \xrightarrow{F} \mathrm{C}$ *such that for any injective object* $I \in \mathrm{A}$, $R^i F(I) = 0$ *for* $i > 0$*. Then there exists a convergent (first quadrant, cohomological) spectral sequence starting on the page* 2*:*

$$E_2^{p,q} = (R^p F)(R^q G)(A) \Rightarrow R^{p+q}(FG)(A). \tag{4.17}$$

For a proof of this, see [Wei94, Sec. 5.8].

4.3 Étale cohomology groups

Speaker: George Robinson

Étale cohomology and Galois cohomology

Let k be a field, $X = \operatorname{Spec} k$ and for the remainder of the section, $G := \operatorname{Gal}(k^{\mathrm{sep}}/k)$ for some fixed separable closure k^{sep} of k.

Recall from theorem 3.12 that $\operatorname{Sh}(X_{\text{ét}})$ is equivalent to $\operatorname{Mod}(G)$, the category of discrete G-modules. Explicitly, for $M \in \operatorname{Mod}(G)$ we have a sheaf $\mathscr{F}_M$ whose sections over a finite separable extension k'/k are given by $M^{G'}$, the elements of M fixed by $G' = \operatorname{Gal}(k^{\mathrm{sep}}/k')$. In the equivalence, the functor $\Gamma(X,-)$ simply becomes the covariant functor $(-)^G \colon \operatorname{Mod}(G) \to \mathrm{Ab}$, which sends a G-module M to the G-invariant submodule M^G. Taking derived functors shows that $H^{\bullet}_{\text{ét}}(X, \mathscr{F}_M) = H^{\bullet}_{\mathrm{Gal}}(K, M)$, that is, the derived functors of $\Gamma(X,-)$ in the étale topology are precisely Galois cohomology.

Example 4.13. With $X = \operatorname{Spec} k$ as above, suppose M is a trivial G-module, that is, M is an abelian group with the trivial action of G, $g \cdot m = m$ for all $g \in G$ and $m \in M$. Then $H^0(G,M) = M$, and from the definition of a cocycle we see that $H^1(G,M) = \operatorname{Hom}(G,M)$.

This is already nontrivial, as for example

$$\begin{aligned} H^1(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}/2\mathbb{Z}) &= \operatorname{Hom}(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}/2\mathbb{Z}) \\ &\cong \{\text{extensions of degree dividing } 2\} \cong \mathbb{Q}^\times/(\mathbb{Q}^\times)^2. \end{aligned} \tag{4.18}$$

An important theorem whose geometric analogue we will encounter later, is the following:

Theorem 4.14 (Hilbert's theorem 90)**.** *If k is a perfect field, then $H^1(G,\overline{k}^\times) = 0$.*

We can apply this to the Kummer sequence (example 3.36) to compute the cohomology of μ_n regarded as a Galois module:

Example 4.15. Applying Galois cohomology to the sequence

$$0 \to \mu_n(\overline{k}) \to \overline{k}^\times \xrightarrow{x \mapsto x^n} \overline{k}^\times \to 0 \tag{4.19}$$

gives the following (rather short) long exact sequence in cohomology,

$$0 \to \mu_n(\overline{k})^G = \mu_n(k) \to k^\times \to k^\times \to H^1(G,\mu_n) \to H^1(G,\overline{k}^\times) = 0, \tag{4.20}$$

where the last equality is Hilbert's theorem 90. We sometimes write this as $H^1(G,\mu_n) = k^\times/(k^\times)^n$. Note that this generalises the previous example, since for $k = \mathbb{Q}$, G acts trivially on $\mu_2 = \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$.

Taking limits in two different ways gives the two identities

$$H^1(G,\mu_{p^\infty}) = k^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p \quad \text{and} \quad H^1(G,\mathbb{Z}_p(1)) = k^\times \widehat{\otimes}\, \mathbb{Z}_p, \tag{4.21}$$

where $\mu_{p^\infty} := \varinjlim_n \mu_{p^n}$ and $\mathbb{Z}_p(1) := \varprojlim_n \mu_{p^n}$.

Example 4.16. Similarly, if E is an elliptic curve over k, then multiplication by p gives an exact sequence

$$0 \to E[p](\overline{k}) \to E(\overline{k}) \to E(\overline{k}) \to 0, \tag{4.22}$$

and the corresponding long exact sequence gives rise to

$$0 \to E(k)/pE(k) \to H^1(G,E[p]) \to H^1(G,E(\overline{k}))[p] \to 0, \tag{4.23}$$

which is the starting point for the definition of Selmer groups, the Tate-Shafarevich group and so on.

Cohomological dimension

Definition 4.17. Let G be a profinite group. We say G has (p-)cohomological dimension at most n if for any (p-)torsion G-module M, we have $H^i(G,M) = 0$ for $i > n$. The (p-)cohomological dimension of G, $\mathrm{cd}(G)$ (resp. $\mathrm{cd}_p(G)$) is the minimal such n.

Given a field k, we tend to write $\mathrm{cd}(k) := \mathrm{cd}(G)$ where $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$.

The following facts are useful in computation. For proofs, see [Ser02, §I.3].

Theorem 4.18. *Let G be any profinite group.*

(i) $\mathrm{cd}(G) = \sup_p cd_p(G)$, *where p runs over the prime numbers.*
(ii) *For any p-Sylow subgroup G_p of G, we have* $\mathrm{cd}_p\, G = \mathrm{cd}_p\, G_p$.[15]
(iii) *Let H be a pro-p group.*[16] *Then* $\mathrm{cd}(H) \le n$ *if and only if* $H^n(H, \mathbb{Z}/p\mathbb{Z}) = 0$.

Corollary 4.19. *For any finite field* $\mathbb{F}_q$*, we have* $\mathrm{cd}(\mathbb{F}_q) = 1$.

Proof. Note that $G = \mathrm{Gal}(\mathbb{F}_q^{\mathrm{sep}}/\mathbb{F}_q) \cong \widehat{\mathbb{Z}}$. The unique p-Sylow subgroup of $\widehat{\mathbb{Z}}$ is $\mathbb{Z}_p$ (since $\widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$) so by (i) and (ii) we get that

$$\mathrm{cd}(\mathbb{F}_q) = \mathrm{cd}(\widehat{\mathbb{Z}}) = \sup_p \mathrm{cd}_p(\mathbb{Z}_p). \tag{4.24}$$

A standard computation shows that $H^0(\mathbb{Z}_p, \mathbb{Z}/p\mathbb{Z}) = H^1(\mathbb{Z}_p, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. On the other hand, $H^2(\mathbb{Z}_p, \mathbb{Z}/p\mathbb{Z})$ classifies isomorphism classes of extensions

$$0 \to \mathbb{Z}/p\mathbb{Z} \to E \to \mathbb{Z}_p \to 0, \tag{4.25}$$

and by looking at the preimage of the toplogical generator of $\mathbb{Z}_p$, it is not too hard to see that E necessarily splits, that is, $H^2(\mathbb{Z}_p, \mathbb{Z}/p\mathbb{Z}) = 0$, whence our result follows. □

A more difficult result is the following:

Theorem 4.20. *If k is a number field and p prime, then*

$$\mathrm{cd}_p(k) = \begin{cases} 2 & \text{if} \quad p > 2 \text{ or } k \text{ is totally imaginary,} \\ \infty & \text{if} \quad p = 2 \text{ and } k \text{ is totally real.} \end{cases} \tag{4.26}$$

The issue here is that a real embedding gives rise to an element of order 2 in the Galois group, and by the Hochschild-Serre spectral sequence we can produce a nontrivial element of $H^i(G, \mathbb{Z}/2\mathbb{Z})$ via $H^i(C_2, \mathbb{Z}/2\mathbb{Z})$, where C_2 is the subgroup generated by complex conjugation. See [Ser02, §II.4.4] for a proof.

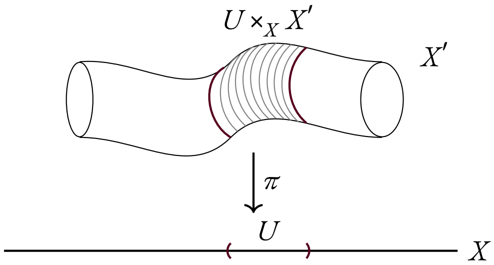
Higher direct images

We now return to cohomology on sites in general. Recall from section 4.1 that given a continuous map of sites $\pi\colon X'_{E'} \to X_E$, the *higher direct images* are $R^n\pi_*(-)$, the right derived functors of the pushforward $\pi_*\colon \mathrm{Sh}(X'_{E'}) \to \mathrm{Sh}(X_E)$.

[15] For G profinite, a p-Sylow subgroup is a subgroup of maximal index not divisible by p (for more detail, see [Ser02, §I.1.4])

[16] A projective limit of p-groups, groups of order a power of p.

Intuitively, higher direct images aim to describe the fibres of the top space X' in terms of the cohomology of the base space. More precisely, if X is a single point, then $R^n\pi_*(-)$ is simply cohomology of the global sections functor. For general X, the idea is that we patch together the cohomology of the fibres X'_x as x varies in X. For example, a theorem of Grothendieck states that under suitably nice conditions, $R^n\pi_*(\mathscr{F})$ vanishes for n greater than the maximal dimension of the fibres.[17]



Proposition 4.21 ([Mil80, III.1.13]). *With π as above, $\mathscr{F} \in \mathrm{Sh}(X'_{E'})$, $R^n\pi_*\mathscr{F}$ is isomorphic to the sheafification of the presheaf*

$$U \mapsto H^n(U \times_X X', \mathscr{F}|_{U \times_X X'}). \tag{4.27}$$

Proof. Recall that $\pi_* = a\pi_p i$, where a is the sheafification functor and $i\colon \mathrm{Sh}(X'_{E'}) \hookrightarrow \mathrm{pSh}(X'_{E'})$ is the inclusion functor. From proposition 3.35, π_p is exact, and so is a by proposition 3.40. The main issue is the failure of exactness of i. Now fix an injective resolution $\mathscr{F} \to \mathscr{I}^\bullet$ – we can always do so since $\mathrm{Sh}(X'_{E'})$ has enough injectives – and note that by exactness,

$$R^n\pi_*\mathscr{F} = H^n(a\pi i I^\bullet) = a\pi_p H^n(iI^\bullet). \tag{4.28}$$

But the presheaf in eq. (4.27) is precisely the one we are applying a to, and this proves our claim. □

With some additional work, one can prove the following stronger result on the small étale site, which states that the cohomology of the fibres is isomorphic to the stalks of the higher direct image functors.

Theorem 4.22 ([Mil80, Thm. III.1.15]). *Let $\pi\colon Y \to X$ be a quasi-compact morphisms of schemes, and $\mathscr{F} \in \mathrm{Sh}(Y_{\text{ét}})$. Let $\bar{x}$ be a geometric point of X, set $i\colon \widetilde{X} :=$ $\operatorname{Spec} \mathcal{O}_{X,\bar{x}} \to X$ and $\widetilde{\mathscr{F}} := i^*\mathscr{F}$. Then*

$$(R^n\pi_*\mathscr{F})_{\bar{x}} \cong H^n(Y \times_X \widetilde{X}, \widetilde{\mathscr{F}}). \tag{4.29}$$

[17] See this M.SE post for a precise statement.

Proof sketch. The idea of the proof is to reduce to the case of U affine, then use what Milne calls a "highly technical result" from EGA which allows us to pass the limit inside the cohomology groups. Then

$$\begin{aligned}(R^n\pi_*\mathscr{F})_{\overline{x}} &\cong \varinjlim_{U} H^n(U\times_X Y, \mathscr{F}|_{U\times_X Y}) \\ &\cong H^n(\varinjlim_{U} U\times_X Y, \varinjlim_{U}\mathscr{F}|_{U\times_X Y}) \\ &= H^n(Y\times_X \widetilde{X}, \widetilde{\mathscr{F}}),\end{aligned} \tag{4.30}$$

where the first isomorphism follows from the preceding proposition, and the second is the highly technical result. □

The following is one of the most famous applications of the Grothendieck spectral sequence:

Theorem 4.23 (Leray spectral sequence)**.** *Let $\pi\colon X'_{E'}\to X_E$ be a continuous map of sites, and $\mathscr{F}\in \mathrm{Sh}(X'_{E'})$. Then we have a spectral sequence beginning on the second page,*

$$H^p(X_E, R^q\pi_*\mathscr{F}) \Rightarrow H^{p+q}(X'_{E'}, \mathscr{F}) \tag{4.31}$$

Proof. We give a quick proof in the case where $X_E = X_{\text{ét}}$, and refer to [Mil80, Thm. III.1.18] for (not much) more detail. For the étale site, π_* has an exact left adjoint so preserves injectives. Thus it satisfies the conditions for the Grothendieck spectral sequence (theorem 4.12), which immediately gives the result. □

4.4 Cohomology with supports

Speakers: George Robinson and George Cooper

One important application of the six functor setup of section 3.5 is to define an algebro-geometric analogue of the theory of cohomology with compact support for manifolds. For the remainder of the section, let $i\colon Z\to X$ be a closed immersion, and $j\colon U = X\setminus Z\to X$ be the corresponding open immersion.

Definition 4.24. The functor

$$\Gamma(Z, i^!(-))\colon \mathrm{Sh}(X_{\text{ét}})\to \mathrm{Ab}, \qquad \mathscr{F}\mapsto \ker\big(\mathscr{F}(X)\to\mathscr{F}(U)\big) \tag{4.32}$$

is left exact, and $H^n_Z(-) := R^n\Gamma(Z, i^!(-))$ is called **the n-th cohomology with support in Z.**

As the name suggests, $H^n_Z(-)$ is a cohomological delta-functor. Cohomology with support in Z relates to usual sheaf cohomology as follows:

Lemma 4.25 ([Mil80, Prop. III.1.25]). *With notation as above, there exists a long exact sequence*

$$\dots \to H^r_Z(X_{\text{ét}}, \mathscr{F}) \to H^r(X_{\text{ét}}, \mathscr{F}) \to H^r(U_{\text{ét}}, \mathscr{F}) \to H^{r+1}_Z(X_{\text{ét}}, \mathscr{F}) \to \dots, \tag{4.33}$$

which is natural in X, Z and $\mathscr{F}$.

Proof. There is a natural isomorphism $\mathrm{Hom}_X(\underline{\mathbb{Z}}_X \mathscr{F}) \xrightarrow{\sim} \Gamma(X_{\text{ét}}, i^!\mathscr{F})$. In particular, $H^p_Z(X_{\text{ét}}, -) \cong \mathrm{Ext}^p_X(\underline{\mathbb{Z}}_X, -)$.

Now, recall that we have an adjunction between j^* and $j_!$, $\mathrm{Hom}_X(j_! j^* \underline{\mathbb{Z}}_X, \mathscr{F}) \cong \mathrm{Hom}_X(j^*\underline{\mathbb{Z}}_X, j^*\mathscr{F})$, so $\mathrm{Ext}_X(j_! h^* \underline{\mathbb{Z}}_X, \mathscr{F}) \cong H^r(U_{\text{ét}}, \mathscr{F})$. We have a short exact sequence (see [Mil80, Rmk. II.3.13]),

$$0 \to j_! j^* \underline{\mathbb{Z}}_X \to \underline{\mathbb{Z}}_X \to i_* i^* \underline{\mathbb{Z}}_X \to 0, \tag{4.34}$$

and since Hom_X is left exact, the sequence

$$0 \to \mathrm{Hom}_X(i_* i^* \underline{\mathbb{Z}}_X, \mathscr{F}) \to \mathrm{Hom}_X(\underline{\mathbb{Z}}_X, \mathscr{F}) \to \mathrm{Hom}_X(j_! j^* \underline{\mathbb{Z}}_X, \mathscr{F}) \tag{4.35}$$

is exact. Therefore we have isomorphisms $\mathrm{Hom}_X(i_* i^* \underline{\mathbb{Z}}_X, \mathscr{F}) \cong \Gamma(X_{\text{ét}}, i^! \mathscr{F})$, so the long exact sequence in $\mathrm{Ext}^\bullet$ gives the corresponding for $H^\bullet_Z$, proving our claim. □

Theorem 4.26 (The Excision Theorem). *Let $\pi \colon X' \to X$ be étale, $Z' \subset X'$ be closed and assume*

(i) *$Z := \pi(Z')$ is closed, and the restriction $\pi|_{Z'} : Z' \to Z$ is an isomorphism;*
(ii) *$\pi(X' \setminus Z') \subset X \setminus Z$.*

Then for any $\mathscr{F} \in \mathrm{Sh}(X_{\text{ét}})$, we have $H^r_Z(X_{\text{ét}}, \mathscr{F}) \cong H^r_{Z'}(X'_{\text{ét}} \mathscr{F}|_{X'})$.

Informally, this says that if we modify X away from Z, the cohomology with support in Z, H^r_Z, is unchanged. This is an algebro-geometric version of a similar statement from algebraic topology.

Proof. By assumption (i), we have a commutative diagram

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d] & X' \arrow[d, "\pi"] & Z' \arrow[l, "i'"'] \arrow[d, "\cong"] \\ U \arrow[r, "j"] & X & Z \arrow[l, "i"'] \end{tikzcd} \tag{4.36}$$

where the undefined objects and maps are the natural ones. From this and the exact sequence in eq. (4.34) (with $\mathscr{F}$ in place of $\underline{\mathbb{Z}}_X$) the diagram

$$\begin{tikzcd} 0 \arrow[r] & \Gamma_{Z'}(X', \pi^*\mathscr{F}) \arrow[r] & \Gamma(X', \pi^*\mathscr{F}) \arrow[r] & \Gamma(U', \pi^*\mathscr{F}) \\ 0 \arrow[r] & \Gamma_Z(X, \mathscr{F}) \arrow[r] \arrow[u, "\phi"] & \Gamma(X, \mathscr{F}) \arrow[r] \arrow[u] & \Gamma(U, \mathscr{F}) \arrow[u] \end{tikzcd} \tag{4.37}$$

commutes. Since π^* is exact and preserves injectives, it suffices to prove the statement for $r = 0$, which amounts to showing that ϕ is an isomorphism.

Let's first prove that ϕ is injective: If $s \in \Gamma_Z(X, \mathscr{F})$ maps to 0, then s restricts to 0 in $\Gamma_Z(X', \mathscr{F})$ and also in $\Gamma(U, \mathscr{F})$, since s is supported on Z. Since $\{X' \to X, U \to X\}$ is an étale cover, $s = 0$ by the sheaf condition.

Next we show surjectivity: if $s' \in \Gamma_{Z'}(X', \pi^*\mathscr{F})$, then the idea is to glue the image of s' in $\Gamma(X', \pi^*\mathscr{F})$ and $0 \in \Gamma(U, \mathscr{F})$ to obtain an element of $\Gamma(X, \mathscr{F})$ which vanishes outside Z, hence pulls back to $\Gamma_Z(X, \mathscr{F})$. But the two agree on $X' \times_X U \subset U'$, and so indeed glue to a global section on X. □

Corollary 4.27. *If $x \in X$ is a closed point and $\mathscr{F} \in \mathrm{Sh}(X_{\text{ét}})$, then we have isomorphisms $H^r_{\{x\}}(X, \mathscr{F}) \cong H^r(\operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,x}, \mathscr{F})$.*

Proof. Apply the theorem to étale neighbourhoods of x, and take the limit using [Mil80, Lemma III.1.16]. □

5 First computations

Contents

5.1 Čech cohomology

Speaker: George Cooper

In this section we restrict our attention to étale sheaves, although several arguments remain true for general sites.

We can develop the machinery of Čech cohomology for the étale topology by analogy with the Zariski case, replacing $U \cap V$ with $U \times_X V$. However, some care must be taken; for example, $U \times_X U \neq U$ in general.

Fix a scheme X, an étale presheaf $\mathscr{F}$ on X, and let $\mathscr{U} = \{U_i \xrightarrow{\phi_i} X\}_{i \in \mathscr{I}}$ an étale cover of X. For $p > 0$ and $i_0, \dots, i_p \in \mathscr{I}$, set $U_{i_0,\dots,i_p} := U_{i_0} \times_X \dots \times_X U_{i_p}$. Then the natural projections onto the factors give rise to a map

$$\mathrm{pr}_{\hat{j}} \colon U_{i_0,\dots,i_p} \to U_{i_0,\dots,\hat{i_j},\dots,i_p}, \qquad 0 \leq j \leq p, \tag{5.1}$$

where $\hat{i}_j$ means that the i_j-component is omitted. Next, let

$$\check{C}^p(\mathcal{U},\mathcal{F}) := \prod_{(i_0,\dots,i_p)} \mathcal{F}(U_{i_0,\dots,i_p}) \tag{5.2}$$

and note that the projections pr_j induce natural maps $\mathrm{res}_j = \mathcal{F}(\mathrm{pr}_j)$, which we use to define

$$d = d^p : \check{C}^p(\mathcal{U},\mathcal{F}) \to \check{C}^{p+1}(\mathcal{U},\mathcal{F}) \quad \text{by} \quad d^p(s_{i_0,\dots,i_p}) \sum_{j=0}^{p+1} (-1)^j \,\mathrm{res}_j(s_{i_0,\dots,\hat{i}_j,\dots,i_{p+1}}). \tag{5.3}$$

One then checks that $d^{p+1}d^p = 0$ for all $p \geq 0$, so $(\check{C}^\bullet(\mathcal{U},\mathcal{F}), d^\bullet)$ forms a complex, and we can take cohomology, giving $\check{H}^p(\mathcal{U},\mathcal{F}) := \ker d^{p+1} / \operatorname{im} d^p$. Then it follows directly from the definition that $\check{H}^0(\mathcal{U},\mathcal{F}) = \Gamma(X_{\text{ét}},\mathcal{F})$.

Note that $\check{C}^\bullet(\mathcal{U},\mathcal{F})$ depends on the choice of covering $\mathcal{U}$; to remove this dependency, we introduce the notion of a *refinement*:

Definition 5.1. A covering $\mathcal{V} = \{V_j \to X\}_{j\in\mathcal{J}}$ is a **refinement of** $\mathcal{U} = \{U_i \to X\}_{i\in\mathcal{I}}$ if there exists a map $\tau\colon \mathcal{I} \to \mathcal{J}$ such that for all $j \in \mathcal{J}$, there exists a map η_j such that triangle

$$\begin{tikzcd} V_j \arrow[rr, "\eta_j"] \arrow[dr] & & U_{\tau(j)} \arrow[dl] \\ & X & \end{tikzcd} \tag{5.4}$$

commutes.

Such a refinement gives maps

$$\tau^p : \check{C}^p(\mathcal{U},\mathcal{F}) \to \check{C}^p(\mathcal{V},\mathcal{F}), \qquad \tau^p(s)_{j_0,\dots,j_p} = \mathrm{res}_{\eta_{j_0}\times\dots\times\eta_{j_p}}(s_{\tau(j_0),\dots,\tau_{j_p}}) \tag{5.5}$$

where $s = s_{i_0,\dots,i_p} \in \check{C}^p(\mathcal{U},\mathcal{F})$. One then checks that $\tau d = d\tau$, so τ induces a map on cohomology, $\rho = \rho(\mathcal{V},\mathcal{U},\tau)$.

Lemma 5.2 ([Mil80, III.2.1])**.** *The map ρ does not depend on the choice of τ and η_j.*

Thus we can talk about a map $\rho = \rho(\mathcal{V},\mathcal{U})\colon \check{H}^\bullet(\mathcal{U},\mathcal{F}) \to \check{H}^\bullet(\mathcal{V},\mathcal{F})$. If $\mathcal{W}$ is a refinement of $\mathcal{V}$, then it is also a refinement of $\mathcal{U}$ (check!) and one can verify that $\rho(\mathcal{W},\mathcal{U}) = \rho(\mathcal{W},\mathcal{V})\rho(\mathcal{V},\mathcal{U})$.

Definition 5.3. The **p-th Čech cohomology group** of $(X,\mathcal{F})$ (for the étale topology) is given by

$$\check{H}^p(X_{\text{ét}},\mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^p(\mathcal{U},\mathcal{F}), \tag{5.6}$$

where the injective limit is taken over the poset of coverings $\mathcal{U}$ with maps ρ as above.

Remark. If $U \to X$ is an étale map and $\mathcal{F}$ a presheaf on the *big* étale site $X_{\text{Ét}}$, then the assignment

$$U \mapsto \check{H}^p(U, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F}) \tag{5.7}$$

where $\mathcal{U}$ runs over coverings of U, is naturally a presheaf on $X_{\text{Ét}}$, denoted $\check{\mathcal{H}}^p(X_{\text{Ét}}, \mathcal{F})$.

Proposition 5.4 ([Mil80, III.2.3-5])**.** *Fix $U \to X$ étale, and an étale covering $\mathcal{U}$ of U.*

(i) *For each $p \geq 0$, $\check{H}^p(\mathcal{U}/U, -) : \mathrm{pSh}(X_{\text{ét}}) \to \mathrm{Ab}$ is isomorphic to the p-th right derived functor of $H^0(\mathcal{U}/U, -)$.*
(ii) *For each $p \geq 0$, $\check{H}^p(U, -) : \mathrm{pSh}(X_{\text{ét}}) \to \mathrm{Ab}$ is isomorphic to the p-th right derived functor of $H^0(U, -)$.*
(iii) *For each $p \geq 0$, $\check{H}^p(X_{\text{ét}}, -) : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Ab}$ is isomorphic to the p-th right derived functor of $\Gamma(X_{\text{ét}}, -)$ if and only if for every $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ there exists a long exact sequence in Čech cohomology.*

Using spectral sequences, we can compute étale cohomology groups from Čech cohomology groups:

Proposition 5.5 ([Mil80, III.2.7])**.** *Let $U \to X$ be étale, $\mathcal{U}$ a covering of U, and $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$. Then we have spectral sequences starting on the second page as follows:*

(i) $\check{H}^p(\mathcal{U}/U, \mathcal{H}^q(\mathcal{F})) \Rightarrow \check{H}^{p+q}(U, \mathcal{F})$.
(ii) $\check{H}^p(\mathcal{U}, \mathcal{H}^q(\mathcal{F})) \Rightarrow \check{H}^{p+q}(U, \mathcal{F})$.

This essentially an application of the Grothendieck spectral sequence, see Milne for more details.

Corollary 5.6. *For any $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ and $U \to X$ étale, there are isomorphisms*

$$\check{H}^0(U, \mathcal{F}) \xrightarrow{\sim} H^0(U, \mathcal{F}) \quad \textit{and} \quad \check{H}^1(U, \mathcal{F}) \xrightarrow{\sim} H^1(U, \mathcal{F}) \tag{5.8}$$

It is natural to ask whether we have isomorphisms in general; under mild conditions this is indeed the case:

Theorem 5.7 ([Mil80, Thm. III.2.17])**.** *Let X be a quasi-compact scheme, and suppose that any finite subset of X is contained in some affine open set.*[18] *Then for every $p \geq 0$ and $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$, we have natural isomorphisms*

$$\check{H}^p(X, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F}). \tag{5.9}$$

The proof is quite technical, and can be found in Milne's book. A further discussion on when derived cohomology and Čech cohomology differ can be found in the following link: MO.

[18] For example, this includes projective schemes over an affine scheme.

5.2 Cohomology of the multiplicative group scheme

Speaker: Mike Daas

In this section, the goal is to compute $H^1(X_{\text{ét}}, \mathbb{G}_m)$ when $X_{\text{ét}}$ is a sufficiently nice scheme of dimension 1. To do so, we first need to recall the language of divisors on schemes:

Let X be a regular integral quasi-compact scheme with function field k, let $g\colon \operatorname{Spec} k \to X$ denote the structure morphism, and denote by $R(U)$ the collection of rational functions on U, for any $U \to X$ is étale. Recall from example 3.36 that $\mathbb{G}_{m,X} := \operatorname{Spec} \mathbb{Z}[t, t^{-1}] \times X$. Note that $R(U)^\times = \Gamma(U, g_* \mathbb{G}_{m,K})$, and the natural map $\Gamma(U, \mathcal{O}_U^\times) \to R(U)^\times$ induces an injection $r\colon \mathbb{G}_{m,K} \to g_* \mathbb{G}_{m,k}$.

Definition 5.8. The sheaf of **Cartier divisors** is the cokernel $\operatorname{Div} X := \operatorname{coker} r = g_* \mathbb{G}_{m,k} / r(\mathbb{G}_{m,k})$.

On the other hand, the notion of a Weil divisor extends naturally to schemes as follows: let X_1 denote the set of points of X of codimension 1. Then all the local rings $\mathcal{O}_{X,x}$ are discrete valuation rings, and we denote by $i_x : \{x\} \hookrightarrow X$ the natural inclusion of a point x into X.

Definition 5.9. The sheaf of **Weil divisors** is the sheaf $D_X := \bigoplus_{x \in X_1} i_{x*} \underline{\mathbb{Z}}$.

Under the conditions above, it is a standard result from scheme theory (eg. [Har77, Prop. II.6.11]) that $D_X \cong \operatorname{Div} X$, and we use the two interchangeably; for example, by definition of $\operatorname{Div} X$, D_X fits into an exact sequence

$$0 \to \mathbb{G}_{m,X} \to \mathbb{G}_{m,k} \to D_X \to 0. \tag{5.10}$$

Using the long exact sequence, we can therefore compute the cohomology of $\mathbb{G}_{m,X}$ in terms of the cohomology of $\mathbb{G}_{m,k}$ and D_X. For the latter, since cohomology commutes with direct sums, it suffices to determine the cohomology of $i_{x*}\underline{\mathbb{Z}}$ for all points $x \in X_1$. The Leray spectral sequence (theorem 4.23) for i_{x*} and $\underline{\mathbb{Z}}$ is

$$H^p(X_{\text{ét}}, R^q i_{x*} \underline{\mathbb{Z}}) \Rightarrow H^{p+q}(x, \underline{\mathbb{Z}}). \tag{5.11}$$

The right hand side is easier to compute explicitly. Let $\kappa(x)$ be the residue field at x and $G_x := \operatorname{Gal}(\kappa(x)^{\text{sep}}/\kappa(x))$. We claim that

$$H^i(x, \underline{\mathbb{Z}}) = \begin{cases} \Gamma(X, \underline{\mathbb{Z}}) = \mathbb{Z} & \text{for} \quad i = 0, \\ 0 & \text{for} \quad i = 1, \\ H^2(x, \mathbb{Z}) \hookrightarrow \operatorname{Hom}_{\text{cts}}(G_x, \mathbb{Q}/\mathbb{Z}) & \text{for} \quad i = 2. \end{cases} \tag{5.12}$$

Here the bottom line means that we can identify $H^2(x, \mathbb{Z})$ with a subgroup of $\operatorname{Hom}_{\text{cts}}(G_x, \mathbb{Q}/\mathbb{Z})$.

By using the equivalence $\mathrm{Sh}(\operatorname{Spec}\kappa(x)_{\text{ét}}) \cong \mathrm{Mod}(G_x)$, we can translate this to a computation in Galois cohomology. Indeed, we have that $H^1(x, \underline{\mathbb{Z}}) \subset \mathrm{Hom}_{\mathrm{cts}}(G_x, \mathbb{Z})$ which vanishes since continuous homomorphisms factor through finite subgroups of $\mathbb{Z}$, and the only such is $\{0\}$; for $i = 2$, we use the long exact sequence arising from

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \tag{5.13}$$

along with the fact that $H^2(G_x, \mathbb{Q}) = 0$ since $\mathbb{Q}$ is an injective object, to show that $H^2(x, \underline{\mathbb{Z}}) \hookrightarrow \mathrm{Hom}_{\mathrm{cts}}(G_x, \mathbb{Q}/\mathbb{Z})$. An application of proposition 4.21 shows that $R^1 i_{x*}\underline{\mathbb{Z}} = 0$.

$$\begin{array}{ccc}
H^0(X_{\text{ét}}, R^2 i_{x*}\underline{\mathbb{Z}}) & H^1(X_{\text{ét}}, R^2 i_{x*}\underline{\mathbb{Z}}) & H^2(X_{\text{ét}}, R^2 i_{x*}\underline{\mathbb{Z}}) \\
 & & \\
H^0(X_{\text{ét}}, R^1 i_{x*}\underline{\mathbb{Z}}) = 0 & H^1(X_{\text{ét}}, R^1 i_{x*}\underline{\mathbb{Z}}) = 0 & H^2(X_{\text{ét}}, R^1 i_{x*}\underline{\mathbb{Z}}) = 0 \\
 & & \\
H^0(X_{\text{ét}}, i_{x*}\underline{\mathbb{Z}}) = \mathbb{Z} & H^1(X_{\text{ét}}, i_{x*}\underline{\mathbb{Z}}) & H^2(X_{\text{ét}}, i_{x*}\underline{\mathbb{Z}})
\end{array}$$

The E_2-page of the spectral sequence $H^p(X_{\text{ét}}, R^q i_{x*}\underline{\mathbb{Z}}) \Rightarrow H^{p+q}(x, \underline{\mathbb{Z}})$.

The spectral sequence[19] implies that

$$H^i(x, i_{x*}\underline{\mathbb{Z}}) = \begin{cases} \Gamma(X, i_{x*}\underline{\mathbb{Z}}) = \mathbb{Z} & \text{for} \quad i = 0, \\ 0 & \text{for} \quad i = 1, \\ H^2(x, i_{x*}\mathbb{Z}) \hookrightarrow \mathrm{Hom}_{\mathrm{cts}}(G_x, \mathbb{Q}/\mathbb{Z}) & \text{for} \quad i = 2. \end{cases} \tag{5.14}$$

and since cohomology commutes with direct sums,

$$H^i(X_{\text{ét}}, D_X) = \begin{cases} \bigoplus_{x \in X_1} \mathbb{Z} & \text{for} \quad i = 0, \\ 0 & \text{for} \quad i = 1, \\ H^2(X_{\text{ét}}, D_X) \hookrightarrow \bigoplus_{x \in X_1} \mathrm{Hom}_{\mathrm{cts}}(G_x, \mathbb{Q}/\mathbb{Z}) & \text{for} \quad i = 2. \end{cases} \tag{5.15}$$

On the other hand, the Leray spectral sequence associated to $g\colon \operatorname{Spec} k \to X$ and $\mathbb{G}_m$ is given by

$$H^p(X_{\text{ét}}, R^q g_* \mathbb{G}_{m,K}) \Rightarrow H^{p+q}(\operatorname{Spec} k, \mathbb{G}_m). \tag{5.16}$$

[19]Explicitly, one argues as follows: In degree 1, we know one of the two terms vanishes, and the other one does not admit nontrivial boundary maps. The result in degree 1 must be zero, so both terms on the E2-page in degree 1 must be zero.

In degree 2, we just include one of the objects in the sequence in degree 2 into the total complex: again no boundary maps reach the term we are looking at and the total complex can be recovered from the degree 2 terms by some kind of filtration.

Recall from theorem 4.22 that the stalk of $R^q g_* \mathbb{G}_{m,k}$ at a geometric point $\overline{x}$ is given by $H^q(\operatorname{Frac}\mathcal{O}_{X,\overline{x}}, \mathbb{G}_m)$. By Hilbert's theorem 90, we know that $H^1(\operatorname{Frac} O_{X,\overline{x}}, \mathbb{G}_m) = 0$, so $R^1 g_* \mathbb{G}_{m,K} = 0$. By a similar argument as for $\underline{\mathbb{Z}}$, by passing to Galois cohomology we find that

$$H^0(X_{\text{ét}}, g_* \mathbb{G}_{m,k}) = H^0(k, \mathbb{G}_m) = k^\times \quad \text{and} \quad H^0(X_{\text{ét}}, g_* \mathbb{G}_{m,k}) \hookrightarrow H^2(k, \mathbb{G}_{m,k}). \tag{5.17}$$

The long exact sequence associated to eq. (5.10) then becomes

$$\begin{aligned} 0 \to \Gamma(X_{\text{ét}}, \mathcal{O}_X^\times) \to k^\times \to \bigoplus_{x \in X_1} i_{x*}\mathbb{Z} \to H^1(X_{\text{ét}}, \mathbb{G}_m) \to 0 \\ 0 \to H^2(X, \mathbb{G}_m) \to H^2(k, \mathbb{G}_{m,k}). \end{aligned} \tag{5.18}$$

In particular, we can identify $H^1(X_{\text{ét}}, \mathbb{G}_m)$ with the quotient $\operatorname{Div}_X/(k^\times/\mathcal{O}_X^\times) = \operatorname{Pic}(X)$.

Stronger results:

Suppose now $\dim X = 1$, and that $\kappa(x)$ is perfect for every $x \in X$. Then $\mathcal{O}_{X,\overline{x}}$ is a Henselian discrete valuation ring with algebraically closed residue field, and we set $k_{\overline{x}} := \operatorname{Frac}\mathcal{O}_{X,\overline{x}}$. We claim that $H^2(k_{\overline{x}}, \mathbb{G}_m) = 0$. Indeed, by [Ser02, II.2.2] we can identify $H^2(k_{\overline{x}}, \mathbb{G}_m)$ with the *Brauer group* $\operatorname{Br}(k_{\overline{x}})$ of $k_{\overline{x}}$, that is, the group of $k_{\overline{x}}$-algebras A with centre equal to $k_{\overline{x}}$, and whose only two-sided ideals are the trivial ones. The group operation is given by the tensor product, $- \otimes_{k_{\overline{x}}} -$. The valuation on $k_{\overline{x}}$ extends uniquely (by Henselian-ness) to a valuation on any $A \in \operatorname{Br}(k_{\overline{x}})$, and we can then produce a subfield $L' \subset A$ with $[A : k_{\overline{x}}] = [L' : k_{\overline{x}}]^2$. $L'/k_{\overline{x}}$ is unramified, so $L' = k_{\overline{x}}$ hence $A = k_{\overline{x}}$. See [Ser95, §XII.2] for more details.

Now, since the stalk of $R^2 g_* \mathbb{G}_{m,k}$ at $\overline{x}$ is given by $H^2(k_{\overline{x}}, \mathbb{G}_m)$, and all stalks vanish, we conclude that $R^2 g_* \mathbb{G}_{m,k} = 0$. The points $x \in X_1$ are all closed, so the functors i_{x*} are all exact by proposition 3.51.

It follows that $H^2(X_{\text{ét}}, g_* \mathbb{G}_{m,k}) = H^2(\operatorname{Spec} k, \mathbb{G}_m)$, and as before we get isomorphisms $H^q(X_{\text{ét}}, i_{x*}\underline{\mathbb{Z}}) \cong H^q(x, \mathbb{Z})$, whence we obtain an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^2(X_{\text{ét}}, \mathbb{G}_m) \arrow[r] & H^2(k, \mathbb{G}_{m,k}) \arrow[r] & \bigoplus_{x \in X_1} \operatorname{Hom}_{\text{cts}}(G_x, \mathbb{Q}/\mathbb{Z}) \arrow[dll] \\ & H^3(X_{\text{ét}}, \mathbb{G}_m) \arrow[r] & H^3(k, \mathbb{G}_{m,k}) & \end{tikzcd} \tag{5.19}$$

If we additionally assume that X is "excellent" (a technical condition we won't define, see [Sta21, Section 07QS] – a scheme is excellent if it can be covered by spectra of excellent rings), then $k_{\overline{x}}$ is quasi-algebraically closed field: any polynomial over it whose number of variables is greater than its degree has a root.

Example 5.10. The Chevalley–Warning theorem ([Ser73, Thm. I.3]) states that any finite field $\mathbb{F}_q$ is quasi-algebraically closed.

With X as above, for any closed point $x \in X$ we have $H^q(k_{\overline{x}}, \mathbb{G}_m) = 0$ hence $R^q g_* \mathbb{G}_m = 0$ for $q > 0$, and this gives the long exact sequence

$$\begin{tikzcd} \dots \arrow[r] & H^r(X_{\text{ét}}, \mathbb{G}_m) \arrow[r] & H^r(k, \mathbb{G}_{m,k}) \arrow[r] & \bigoplus_{x \in X_1} H^{r-1}(G_x, \mathbb{Q}/\mathbb{Z}) \arrow[dlll, out=0, in=180] \\ H^{r+1}(X_{\text{ét}}, \mathbb{G}_m) \arrow[r] & H^{r+1}(k, \mathbb{G}_{m,k}) \arrow[r] & \dots \end{tikzcd} \tag{5.20}$$

For example, if X is a smooth algebraic curve over an algebraically closed field, then k is C_1 by Tsen's theorem ([Sta21, Theorem 03RD]). Since we also have that $H^r(G_x, \mathbb{Q}/\mathbb{Z}) = 0$ for $r \geq 1$, as $k_{\overline{x}}$ is algebraically closed, we deduce from the long exact sequence that $H^r(X_{\text{ét}}, \mathbb{G}_m) = 0$ for all $r \geq 2$.

5.3 Comparing topologies

In this section, we take a brief pause from the gritty computations to answer the question "how do we compare cohomology groups defined with respect to different Grothendieck topologies on a site?" We don't give any proofs, but refer the eager reader to Milne's book.

Proposition 5.11 ([Mil80, Prop. III.3.1]). *Let $(C'/X)_E$ be a Grothendieck topology, let $C \subset C'$ be a subcategory and suppose $f\colon (C'/X)_E \to (C/X)_E$ is the map induced by the inclusion functor $C \hookrightarrow C'$. For any sheaves $\mathscr{F}'$ on $(C'/X)_E$ and $\mathscr{F}$ on $(C/X)_E$, we have*

$$H^i(X, f_*\mathscr{F}') \cong H^i(X, \mathscr{F}') \quad \text{and} \quad H^i(X, \mathscr{F}) \cong H^i(X, f^*\mathscr{F}'), \tag{5.21}$$

for all $i \geq 0$.

In particular, we can pass freely between the small and big étale sites when computing étale cohomology groups.

Definition 5.12. Let $(C_1/X)_{E_1}$ and $(C_2/X)_{E_2}$ be sites where $C_1 \supset C_2$ and $E_1 \supset E_2$. If for every covering in the E_1-topology there exists a covering in E_2 which refines it and vice versa, then we say that E_1 and E_2 admit **mutual refinements**.

Proposition 5.13 ([Mil80, Prop. III.3.3]). *Suppose E_1 and E_2 as above are* stable classes, *cf. definition 2.4. Let $f\colon (C_1/X)_{E_1} \to (C_2/X)_{E_2}$ be the natural map. If E_1 and E_2 admit mutual refinements, then*

$$H^i(X_{E_2}, f_*\mathscr{F}) \cong H^i(X_{E_1}, \mathscr{F}), \tag{5.22}$$

for any sheaf $\mathscr{F} \in \mathrm{Sh}(X_{E_1})$ and $i \geq 0$.

We can use this to restrict from the class of étale morphisms to the class of étale morphisms of finite type, or to the class of separated étale morphisms, or even to affine étale morphisms (exercise! – this amounts to showing that suitable mutual refinements exist). In a similar fashion we can reduce a problem from the class of smooth morphisms to the class of étale morphisms: the key point in showing this is that every smooth morphism admits a section étale-locally. The following shows that we can also restrict our attention to finite subcoverings:

Proposition 5.14 ([Mil80, Prop. III.3.5])**.** *Suppose $(C/X)_E$ is a* Noetherian site, *meaning that every covering has a finite subcovering, i.e. a covering consisting of finitely many elements. Let E_f denote the category of finite subcoverings. Then the categories of sheaves (resp. presheaves) on X_E and X_{E_f} are canonically equivalent. In particular, cohomology is preserved when passing from one to the other.*

5.4 Étale and complex cohomology

In this section, for a scheme X over $\operatorname{Spec}\mathbb{C}$, we let $H^i(X(\mathbb{C}),-)$ denote the usual singular cohomology. If étale cohomology is indeed a "good" cohomology theory, then it should coincide with singular cohomology under suitably nice conditions. The goal of this section is to prove the following theorem:

Theorem 5.15. *Let X be a smooth scheme over* $\operatorname{Spec}\mathbb{C}$ *and M a finite abelian group. Then*

$$H^i(X(\mathbb{C}),M) \cong H^i(X_{\text{ét}},\underline{M}), \tag{5.23}$$

for all $i \geq 0$.

Example 5.16. Note that it is crucial to assume M is finite; for example, if X is an elliptic curve, we have

$$H^1(X(\mathbb{C}),\mathbb{Z}) = \mathbb{Z}^2 \quad \text{but} \quad H^1(X_{\text{ét}},\mathbb{Z}) = \operatorname{Hom}_{\text{cts}}(\pi_1(X),\mathbb{Z}) = 0, \tag{5.24}$$

the latter because $\pi_1(X)$ is a profinite group, by the same argument as in section 5.2.

Proof (sketch). For $i = 0$, this amounts to showing that the numbers of components of X and $X(\mathbb{C})$ agree. This follows from a reduction to the case of X being a projective curve, and then an appeal to Riemann-Roch. For details, see [Sha13, §VII.2].

For $i = 1$, we use the fact that $H^1(X_{\text{ét}},\underline{M})$ is in bijective correspondence with the set of Galois coverings with automorphism group equal to M; see [Mil00, Props. 11.1 & 11.3]. On the other hand, since $H^1(X(\mathbb{C}),M)$ classifies analytic covering spaces with automorphism group M, the case follows from the following theorem:

Theorem 5.17 (Riemann existence theorem)**.** *Let $X \to \operatorname{Spec}\mathbb{C}$ be locally of finite type. Then there is an equivalence of categories*

$$\begin{aligned}\{\textit{finite étale covers } Y \to X\} \leftrightarrow \{&\textit{analytic covering spaces } Y^{\mathrm{an}} \to X^{\mathrm{an}}\},\\ &Y \mapsto Y^{\mathrm{an}}.\end{aligned} \tag{5.25}$$

Finally, for $i > 1$, let X_{cx} be the site on X^{an} where coverings are given by local isomorphisms of complex analytic spaces. Since for any complex-open $U \subset X(\mathbb{C})$ the map $U \hookrightarrow X(\mathbb{C})$ is a local isomorphism, we have a natural map $X_{\mathrm{cx}} \to X(\mathbb{C})_{\mathrm{top}}$, where $X(\mathbb{C})_{\mathrm{top}}$ denotes the site generated by the complex topology of $X(\mathbb{C})$. It is not difficult to see that these admit mutual refinements, so by proposition 5.14,

$$H^i(X_{\mathrm{cx}}, \underline{M}) \cong H^i(X_{\mathrm{top}}, \underline{M}). \tag{5.26}$$

By the implicit function theorem (cf. exercise sheet 1), for $U \to X$ étale that associated map $U^{\mathrm{an}} \to X^{\mathrm{an}}$ is a local isomorphism, giving rise to a map of sites $f\colon X_{\mathrm{cx}} \to X_{\text{ét}}$. This gives rise to a Leray spectral sequence

$$H^i(X_{\text{ét}}, R^j f_* \mathscr{F}) \Rightarrow H^{i+j}(X_{\mathrm{cx}}, \mathscr{F}). \tag{5.27}$$

If we can show that $R^j f_* \mathscr{F} = 0$ for $j > 0$, then the spectral sequence degenerates and we are done. By proposition 4.21, $R^j f_* \mathscr{F}$ is the sheafification of $U \mapsto H^j(U_{\mathrm{cx}}, \mathscr{F})$. The final ingredient in the proof is the following lemma, which relies on quite heavy machinery, namely [Mil80, VI.4.2 & 5.1]. We refer the reader to Milne's book for a proof.

Lemma 5.18 ([Mil80, III. 3.15])**.** *For a locally constant sheaf $\mathscr{F} \in \operatorname{Sh}(X_{\mathrm{cx}})$ with finite fibres and $i > 0$, fix $\gamma \in H^i(X_{\mathrm{cx}}, \mathscr{F})$. For any $x \in X(\mathbb{C})$, there exists an étale morphism $U \to X$ whose image contains x with $\gamma|_{U_{\mathrm{cx}}} = 0$.*

In particular, for our constant sheaf $\underline{M}$, we have that $H^i(U_{\mathrm{cm}}, \underline{M}) = 0$ and upon sheafifying this gives that $R^j f_* \mathscr{F} = 0$, as required. □

Mount Étale in Savoie, France[20]

6 Cohomology of curves

Contents

6.1 Constructible sheaves

Speaker: Håvard Damm-Johnsen

In this section, we assume as always that all schemes are locally Noetherian, and sheaves are assumed to be valued in Ab or $\mathrm{Mod}(\underline{\mathbb{Z}/n\mathbb{Z}})$, although most of the results extend to sheaves valued in modules over an arbitrary Noetherian ring.

Key motivation: we want a "nice" category for coefficient systems of schemes. Issue: the category of locally constant sheaves is not well-behaved, in particular, not closed under pushforward.

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Example 6.1. Let G be a finite abelian group, and let $i\colon 0 \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$ be the inclusion of $0 = \operatorname{Spec}\mathbb{C}$ corresponding to the origin. Then $i_*\underline{G}$ is the skyscraper sheaf, and is not locally constant: if $U \to \mathbb{A}^1_{\mathbb{C}}$ is etale with 0 in its image, then the stalk of $i_*\underline{G}$ at 0 is different from the stalk away from 0 so i_*G not constant on any etale covering.

Recall that a sheaf $\mathscr{F}$ is *locally constant* on a scheme X if there exists some étale covering $\{\phi : U \to X\}$ such that $\mathscr{F}|_U := \phi^*\mathscr{F}$ is a constant sheaf.

We first define constructible sheaves on a Noetherian scheme:

Definition 6.2. Let X be a Noetherian scheme. A sheaf $\mathscr{F} \in \operatorname{Sh}(X_{\text{ét}})$ is **constructible** if there exists a finite partition $X = \bigsqcup_i Z_i$ where Z_i are locally closed subschemes of X, and $\mathscr{F}|_{Z_i}$ is locally constant with finite stalks.

We frequently shorten "locally constant with finite stalks" to "finite locally constant". The reason for restricting our attention to such sheaves is that cohomology with infinite coefficient sheaves is frequently ill-behaved, as in example 5.16.

Remark. In [Mil80, §V.1], Milne defines constructible sheaves via algebraic spaces, while we loosely follow the approach of [Sta21, Section 05BE], albeit in lesser generality.

Constructible sheaves extend the class of locally constant sheaves by allowing them to vary along closed subschemes. We can think of these as "locally locally constant sheaves", and by taking the trivial partition it is clear that any locally constant sheaf with finite stalks is constructible.

Lemma 6.3. *Let X be Noetherian. Then we can check constructibility Zariski-locally.*

Proof. We want to show that if $\mathscr{F}|_{U_i}$ is constructible for some Zariski-open covering $X = \bigcup_i U_i$, then $\mathscr{F}$ is constructible. Since X is quasi-compact, we can assume $\{U_i\}$ is finite. If $U_i = \sqcup_j Z_{ij}$ with $\mathscr{F}_{Z_{ij}}$ locally constant with finite stalks, then we have a decomposition $X = \bigcup_{ij} Z_{ij}$. By the usual topological argument, this can be refined to a disjoint union $X = \sqcup_{i'} Z'_{i'}$ with $\mathscr{F}|_{Z'_{i'}}$ finite locally constant. □

This allows us to extend the definition of constructibility to arbitrary locally Noetherian schemes in a natural way.

Proposition 6.4 ([Sta21, Tag 095H]). *Let $f : X' \to X$ be a finite étale morphism, and $\mathscr{F}' \in \operatorname{Sh}(X'_{\text{ét}})$ a constructible sheaf. Then $f_*\mathscr{F}'$ is also constructible.*

This is somewhat technical, and in the interest of time we won't go into details. The full subcategory of $\operatorname{Sh}(X_{\text{ét}})$ consisting of constructible sheaves retains several good properties:

Theorem 6.5. *The category of constructible sheaves is closed under closed under taking kernels, cokernels, extensions and tensor products, and is abelian.*

Tate twists

Tate twists are a nifty device for stating Poincaré duality without making a choice of an orientation. See this link for a less vague explanation.

Fix now a scheme X such that n is invertible in every residue field of X. Then we saw in exercise sheet 3 that μ_n is locally isomorphic to the constant sheaf $\underline{\mathbb{Z}/n\mathbb{Z}}$, and we regard μ_n as a locally free sheaf of $\underline{\mathbb{Z}/n\mathbb{Z}}$-modules of rank 1 on $X_{\text{ét}}$.

Lemma 6.6. *If $\mathcal{F}$ is a locally free and constructible sheaf with values in $\underline{\mathbb{Z}/n\mathbb{Z}}$, then so is its dual, $\mathcal{F}^\vee := \mathcal{H}om_{\underline{\mathbb{Z}/n\mathbb{Z}}}(\mathcal{F}, \underline{\mathbb{Z}/n\mathbb{Z}})$.*

Proof. Choose an étale covering $\{U \to X\}$ such that $\mathcal{F}|_U$ is free; then $\mathcal{F}^\vee|_U$ is also free. By lemma 6.3 we can check constructibility locally, where it is immediate. □

In particular, we can consider the dual of μ_n. Let

$$(\underline{\mathbb{Z}/n\mathbb{Z}})(r) := \begin{cases} \mu_n^{\otimes r} & \text{if} \quad r > 0, \\ \underline{\mathbb{Z}/n\mathbb{Z}} & \text{if} \quad r = 0, \\ \left(\mu_n^{\otimes(-r)}\right)^\vee & \text{if} \quad r < 0. \end{cases} \tag{6.1}$$

This is a sheaf of $\underline{\mathbb{Z}/n\mathbb{Z}}$-modules by the previous lemma.

Definition 6.7. Let $\mathcal{F}$ be a sheaf of $\underline{\mathbb{Z}/n\mathbb{Z}}$-modules, and fix $r \in \mathbb{Z}$. The **r-th Tate twist of $\mathcal{F}$** is $\mathcal{F}(r) := \mathcal{F} \otimes (\underline{\mathbb{Z}/n\mathbb{Z}})(r)$.

Proposition 6.8. *Let $\mathcal{F}$ be a constructible sheaf. Then $\mathcal{F}(r)$ is locally isomorphic to $\mathcal{F}$.*

6.2 Poincaré duality

A very readable introduction to this is Tony Feng's notes.

The intuition for Poincaré duality is most easily seen in the case of a real compact n-manifold M. Recall that for each for each $0 \le k \le n$, the *cup product*

$$H^k(M;\mathbb{R}) \times H^{n-k}(M;\mathbb{R}) \to H^n(M;\mathbb{Z}) \tag{6.2}$$

defines a non-degenerate bilinear map.

Theorem 6.9 (Classical Poincaré duality)**.** *Let M be an orientable compact manifold of real dimension n. Then a choice of an orientation on M defines a* trace map

$$\int_M : H^n(M;\mathbb{R}) \to \mathbb{R}, \tag{6.3}$$

which in turn gives an identification

$$H^i(M;\mathbb{R}) \cong \left(H^{n-i}(M;\mathbb{R})\right)^\vee \cong H_{n-i}(M;\mathbb{R}). \tag{6.4}$$

Since we don't have a canonical way of orienting our schemes, we ought to study the above when M is not oriented. In that case we need to add extra conditions on the coefficient ring; for example, we always have Poincaré duality for cohomology with coefficients in $\mathbb{Z}/2\mathbb{Z}$. This is done by introducing an *orientation sheaf*, whose analogue in the sheaf setting is $\mu^{\otimes r}$. Postponing some essential definitions, we give the statement of Poincare duality for algebraic curves:

Theorem 6.10 (Poincaré duality for curves, [Mil80, Thm. V.2.1])**.** *Let X be a smooth projective curve over an algebraically closed field k, and suppose $n \in \mathbb{Z}$ is invertible in k.*

(a) If $U \subset X$ is a non-empty open subscheme, then there is a canonical isomorphism

$$\eta(U) : H_c^2(U, \mu_n) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}. \tag{6.5}$$

(b) For any constructible sheaf $\mathscr{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $U_{\text{ét}}$, the groups $H_c^r(U, \mathscr{F})$ and $\operatorname{Ext}^r_{\operatorname{Sh}(U_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})}(\mathscr{F}, \mu_n)$ are finite for all r and vanish for $r > 2$. The pairing

$$H_c^r(U, \mathscr{F}) \times \operatorname{Ext}^r_{\operatorname{Sh}(U_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})}(\mathscr{F}, \mu_n) \to H_c^2(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z} \tag{6.6}$$

is non-degenerate.

Remark. The assumption that k be algebraically closed is necessary; however there are analogues for other fields, for example Tate-Poitou duality in Galois cohomology, and more generally Artin-Verdier duality for $\operatorname{Spec} \mathcal{O}_K$, when K is a number field. There are also many generalisations of Poincaré duality, in particular Verdier duality, see for example [KS13].

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Let us first explain the cohomology groups H_c:

Definition 6.11. Let $j : U \hookrightarrow X$ be an open immersion, and $\mathscr{F} \in \operatorname{Sh}(U_{\text{ét}})$. Then $H_c^\bullet(U, \mathscr{F}) := H^\bullet(X, j_!\mathscr{F})$ is called **cohomology with compact support**.

We can define this more generally for $U \to \operatorname{Spec} k$ separated of finite type: by Nagata compactification ([Sta21, Theorem 0F41] or Brian Conrad's notes), it factors as

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr] & X \arrow[d] \\ & \operatorname{Spec} k \end{tikzcd} \tag{6.7}$$

where j is an open immersion and $X \to \operatorname{Spec} k$ is proper, and one can check that $H_c^\bullet(U, \mathscr{F})$ is independent of the choice of compactification X. One can also check that given a short exact sequence of sheaves on U, there is a corresponding long exact sequence in $H_c^\bullet$.

Next, let's define the pairing of eq. (6.6):[21] for transparency, fix an abelian category $\mathscr{A}$ with enough injectives, and let A, B and C be objects in $\mathscr{A}$. It is well-known (see e.g. Wikipedia) that $\mathrm{Ext}^r(A,B)$ classifies "r-extensions"

$$0 \to B \to X_r \to \ldots \to X_1 \to A \to 0 \tag{6.8}$$

up to equivalence, and if

$$\xi = 0 \to B \to X_r \to \ldots \to X_1 \to A \to 0 \quad \text{and} \quad \xi' = 0 \to C \to X'_s \to \ldots \to X'_1 \to B \to 0$$

are elements of $\mathrm{Ext}^r(A,B)$ and $\mathrm{Ext}^s(B,C)$ respectively, then there is a natural pairing

$$\begin{array}{ccc} \mathrm{Ext}^n(A,B) \times \mathrm{Ext}^m(B,C) & \to & \mathrm{Ext}^{n+m}(A,C) \\ (\xi, \xi') & \mapsto & \xi \smile \xi', \end{array} \tag{6.9}$$

where

$$\xi \smile \xi' = 0 \to C \to X'_s \to \ldots \to X'_1 \to X_s \to \ldots \to X_1 \to A \to 0, \tag{6.10}$$

the map $X'_1 \to X_r$ being the natural composition $X'_1 \to B \to X_r$. One then checks that $\xi \smile \xi'$ is actually an extension, hence a well-defined element of $\mathrm{Ext}^{r+s}(A,C)$.

Returning to our situation, we take $\mathscr{A} = \mathrm{Sh}(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})$, $A = \underline{\mathbb{Z}}$, $B = j_!\mathscr{F}$ and $C = j_!\mu_n$. Since Ext are the derived functors of Hom, we have $A = \mathrm{Ext}^r(\underline{\mathbb{Z}}, j_!\mathscr{F}) \cong H^r_c(U,\mathscr{F})$ and $C = \mathrm{Ext}^{r+s}(\underline{\mathbb{Z}}, j_!\mu)_n \cong H^{r+s}_c(U,\mu_n)$, and by the adjunction $(j^*, j_!)$ we find

$$B = \mathrm{Ext}^s(j_!\mathscr{F}, j_!\mu_n) \cong \mathrm{Ext}^s(\mathscr{F}, j^* j_!\mu_n) \cong \mathrm{Ext}^s(\mathscr{F}, \mu_n). \tag{6.11}$$

Altogether, this gives the pairing in eq. (6.6). Now we are ready to sketch the main ideas of the proof of theorem 6.10.

Proof of a). We want to construct a canonical isomorphism $\eta(U) \colon H^2_c(U,\mu_n) \cong \mathbb{Z}/n\mathbb{Z}$. First assume $U = X$. Taking the long exact sequence in $H^\bullet_c = H^\bullet$ associated to the Kummer sequence (example 3.36), we get

$$\ldots \to H^1(X,\mathbb{G}_m) \xrightarrow{\cdot n} H^1(X,\mathbb{G}_m) \to H^2(X,\mu_n) \to H^2(X,\mathbb{G}_{m)} \to H^2(X,\mathbb{G}_m) \to \ldots \tag{6.12}$$

and we make a few observations:

- $H^m(X,\mathbb{G}_m) = 0$ for all $m \geq 2$ by Tsen's theorem.
- $H^1(X,\mathbb{G}_m) \cong \mathrm{Pic}(X)$ as in section 5.2.
- There is a natural short exact sequence $0 \to \mathrm{Jac}_k X \to \mathrm{Pic}(X) \to \mathbb{Z} \to 0$, where $\mathrm{Jac}_k(X)$ is the *Jacobian* of X.

[21] I found the explanation on Wikipedia a lot more intuitive than the one from the talk, so I decided to type up that.

- There is a surjective multiplication map $\mathrm{Jac}_k X \to \mathrm{Jac}_k X$.

Therefore we have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathrm{Jac}_k(X) \arrow[r] \arrow[d] & \mathrm{Jac}_k(X) \arrow[r] \arrow[d] & 0 & \\
 & H^1(X,\mathbb{G}_m) \arrow[r] \arrow[d] & H^1(X,\mathbb{G}_m) \arrow[r] \arrow[d] & H^2(X,\mu_n) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot n"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd} \tag{6.13}$$

and the snake lemma applied to the two bottom rows implies that the induced dashed map $H^2(X,\mu_n) \to \mathbb{Z}/n\mathbb{Z}$ is an isomorphism. For general $U \hookrightarrow X$, let $i\colon Z := X \setminus U \to Z$ be the natural inclusion, and recall that we have a short exact sequence

$$1 \to j_! j^* \mu_n \to \mu_n \to i_* i^* \mu_n \to 1, \tag{6.14}$$

inducing a long exact sequence

$$\dots \to H^p(X, j_! j^* \mu_n) \cong H^p_c(U,\mu_n) \to H^p(X,\mu_n) \to H^p(X, i_* i^* \mu_n) \to H^{p+1}(X, j_! j^* \mu_n) \to \dots . \tag{6.15}$$

We claim that $H^\bullet(X, i_* i^* \mu_n) = 0$: since $H^\bullet(X, i_* i^* \mu_n) = H^\bullet(Z, i^* \mu_n)$, and Z consists of a finite collection of points, $Z = \bigsqcup \operatorname{Spec} k$, Tsen's theorem implies that $H^\bullet(Z, i^* \mu_n) = 0$. Exactness of the long exact sequence and the proof for $U = X$ then gives $H^p_c(U,\mu_n) \cong H^p(X,\mu_n) \cong \mathbb{Z}/n\mathbb{Z}$ for all p. □

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