# Étale cohomology reading seminar

Exercises for Week 6

**Exercise 1.** Let $i : Z \to X$ and $j : U \to X$ be complementary closed and open immersions, respectively.

(i) Let $\mathscr{F}$ be an étale sheaf on $U$. Define a short exact sequence (natural in $\mathscr{F}$)

$$0 \to j_!\mathscr{F} \to j_*\mathscr{F} \to C_{\mathscr{F}} \to 0$$

and express the cokernel $C_{\mathscr{F}}$ in terms of the 'six functors' from class.

(ii) Compute $C_{\mathscr{F}}$ explicitly in the following situation(s):

- *(for the geometrically minded)* Let $X = \mathbb{A}^1_{\mathbb{C}}$ be the complex affine line, and $i : Z = \mathrm{Spec}(\mathbb{C}) \to X$ the origin. Let $\mathscr{F} = \mathscr{O}_U$ be the structure sheaf on $U = \mathbb{A}^1 \backslash \{0\}$.
- *(for the arithmetically minded)* Let $X = \mathrm{Spec}(\mathbb{Z}_{(p)})$ be the spectrum of the local ring of $\mathbb{Z}$ at some prime $p$, let $i : Z = \mathrm{Spec}(\mathbb{F}_p) \to X$ be the closed point, and $\mathscr{F} = \mathscr{O}_U$ the structure sheaf on $U = \mathrm{Spec}(\mathbb{Q}_p)$.

(iii) Assume $\mathscr{F} = j^*\mathscr{G}$ is the restriction of some sheaf $\mathscr{G}$ on $X$. How does $\mathscr{G}$ compare to $j_*\mathscr{F}$? It's useful to attempt an answer in this generality but you might also want to consider the particular situations of the previous part.

**Exercise 2** (Optional)**.** (Milne, Exercise II.3.7) Let $X$ be an integral scheme with generic point $g\colon \eta \to X$.

(i) Show that if $X$ is normal, then $g_*M_\eta = M_X$ for any constant sheaf $M_\eta$ on $\eta$.

(ii) Show that if $X$ is a curve with a node $i\colon z \hookrightarrow X$, then there is an exact sequence,

$$0 \to M_X \to g_*M_\eta \to i_*M_z \to 0. \qquad (1)$$

What is true in general? (Hint: write $g$ as the composite $\eta \to X' \to X$ where $X'$ is the normalisation of $X$.)