Étale cohomology reading seminar

Exercises for Week 3

Exercise 1. Let X be a scheme and we consider sheaves on the small étale site of X. Suppose that the characteristic of the residue field to any point of X does not divide n. Show then that the sheaves μ_n and $\mathbb{Z}/n\mathbb{Z}$ are locally isomorphic[1].

As we didn't have time to discuss the second exercise from last week, we put it here too, aiming to talk about it next time.

Exercise 2 (Repeated from previous week)**.** Let X be a Noetherian and connected scheme and let $f\colon Y \to X$ be a finite flat morphism. By the previous exercise, we have that $f_*\mathcal{O}_Y$ is locally free, of constant rank r, say. Show:

(i) There is a sheaf of ideals $\mathfrak{D}_{Y/X}$ on X, called the discriminant of Y over X, with the property that if U is an open affine in X such that $B = \Gamma(f^{-1}(U), \mathcal{O}_Y)$ is free with basis $(b_1, \dots, b_r)$ over $A = \Gamma(U, \mathcal{O}_X)$, then $\Gamma(U, \mathfrak{D}_{Y/X})$ is the principal ideal generated by $\det\big(\mathrm{Tr}_{B/A}(b_i b_j)\big)$.[2]

(ii) f is unramified, hence étale, at all $y \in f^{-1}(x)$ if and only if $(\mathfrak{D}_{Y/X})_x = \mathcal{O}_{X,x}$.

(iii) If f is unramified at all $y \in f^{-1}(x)$ for some x, then there is some open subset $U \subset X$ containing x such that $f\colon f^{-1}(U) \to U$ is étale.

(iv) If $B = A[T]/(P(T))$ with P monic, then the discriminant is $\mathfrak{D}_{B/A} = (D(P))$, where $D(P)$ is the discriminant of P, that is, the resultant, $\mathrm{res}(P, P')$ of P and P'. Show also that the different[3] $\mathfrak{d}_{B/A} = (P'(t))$, where $t = T \pmod{P}$.

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[1] By $\mathbb{Z}/n\mathbb{Z}$ we mean the sheafification of the constant presheaf with value $\mathbb{Z}/n\mathbb{Z}$. We define the sheaf μ_n to be the kernel of the rising to the nth power map $\mathbb{G}_m \to \mathbb{G}_m$. Recall that for $Y \to X$ étale, we have $\mathbb{G}_m(Y) = \mathcal{O}_Y(Y)^\times$, and that the rising to the nth power map is defined over Y as $s \mapsto s^n$, for $s \in \mathbb{G}_m(Y)$. Note that μ_n is representable by the scheme $\mathrm{Spec}(\mathbb{Z}[t, t^{-1}]/(t^n - 1)) \times X$.

[2] If $b \in B$, $\mathrm{Tr}_{B/A}(b)$ denotes the trace of the multiplication by b A-linear map $B \to B$. As B is a finite free A-module, the trace of an A-linear endomorphism of B is defined. We have the trace form $B \times B \to A\colon (b, b') \mapsto \mathrm{Tr}_{B/A}(bb')$, which is symmetric and A-bilinear, and if (b_i) is a basis of B as an A-module, then $(\mathrm{Tr}_{B/A}(b_i b_j))$ is the matrix of the trace form in that basis.

[3] The *different* $\mathfrak{d}_{Y/X}$ of a morphism $f\colon Y \to X$ locally of finite type is the annihilator of $\Omega^1_{Y/X}$, which is an ideal sheaf of $\mathcal{O}_Y$.