Étale cohomology reading seminar

Exercise sheet 10

Exercise 1. Serre [Ser64] has constructed a smooth projective variety V over a number field K and two complex embeddings $\phi, \psi : K \to \mathbb{C}$ such that the associated topological spaces

$$V_\phi = \left(V \times_{K,\phi} \mathbb{C}\right)(\mathbb{C}), \qquad V_\psi = \left(V \times_{K,\psi} \mathbb{C}\right)(\mathbb{C})$$

have non-isomorphic fundamental groups, in particular are not homeomorphic. Show that V_ϕ and V_ψ *do* have the same Betti numbers (using—what else?—étale cohomology).

Exercise 2. Let k be a field which contains all n-th roots of unity, where $n \in \mathbb{Z}_{>0}$ is invertible in k. Let X be a k-variety, and F be an étale sheaf of $\mathbb{Z}/n$-modules on X. Show that for every $p \geq 0$ and $r \in \mathbb{Z}$ there is a *canonical* isomorphism of Galois modules

$$H^p(X, F(r)) \cong H^p(X, F)(r).$$

Bibliography

[Ser64] Jean-Pierre Serre. Exemples de variétés projectives conjuguées non homéomorphes. *C. R. Acad. Sci. Paris*, 258:4194–4196, 1964.