Étale cohomology reading seminar

Exercises for Week 1

Exercise 1 (Milne, Theorem 2.9)**.** Let A be a Noetherian ring and let M be a finitely generated A-module. The following are equivalent:

(i) M is flat;
(ii) M_m is a free A_m-module for all maximal ideals m of A;
(iii) $\widetilde{M}$ is a locally free sheaf on $\operatorname{Spec}(A)$;
(iv) M is a projective A-module.

Moreover, if A is an integral domain, they are equivalent to:

(vi) $\dim_{\kappa)(P)}(M \otimes_A \kappa(P))$ is the same for all prime ideals P of A.

Use this fact to prove that if $f\colon Y \to X$ is a finite morphism of schemes with X Noetherian, then f is flat if and only if $\mathscr{F} = f_*\mathcal{O}_Y$ is locally free. If X is also integral, then this is equivalent to the function $X \to \mathbb{N}\colon x \mapsto \dim_{\kappa(x)}(\mathscr{F}_x \otimes_{\mathcal{O}_x} \kappa(x))$ being constant.

Exercise 2 (Milne, exercise 3.9)**.** Let X be a Noetherian and connected scheme and let $f\colon Y \to X$ be a finite flat morphism. By the previous exercise, we have that $f_*\mathcal{O}_Y$ is locally free, of constant rank r, say. Show:

(i) There is a sheaf of ideals $\mathfrak{D}_{Y/X}$ on X, called the discriminant of Y over X, with the property that if U is an open affine in X such that $B = \Gamma(f^{-1}(U), \mathcal{O}_Y)$ is free with basis $(b_1, \ldots, b_r)$ over $A = \Gamma(U, \mathcal{O}_X)$, then $\Gamma(U, \mathfrak{D}_{Y/X})$ is the principal ideal generated by $\det\big(\operatorname{Tr}_{B/A}(b_i b_j)\big)$.
(ii) f is unramified, hence étale, at all $y \in f^{-1}(x)$ if and only if $(\mathfrak{D}_{Y/X})_x = \mathcal{O}_{X,x}$.
(iii) If f is unramified at all $y \in f^{-1}(x)$ for some x, then there is some open subset $U \subset X$ containing x such that $f\colon f^{-1}(U) \to U$ is étale.
(iv) If $B = A[T]/(P(T))$ with P monic, then the discriminant is $\mathfrak{D}_{B/A} = (D(P))$, where $D(P)$ is the discriminant of P, that is, the resultant, $\operatorname{res}(P, P')$ of P and P'. Show also that the different[1] $\mathfrak{d}_{B/A} = (P'(t))$, where $t = T \pmod{P}$.

[1]The *different* $\mathfrak{d}_{Y/X}$ of a morphism $f\colon Y \to X$ locally of finite type is the annihilator of $\Omega^1_{Y/X}$, which is an ideal sheaf of $\mathcal{O}_Y$