Étale cohomology reading seminar

Håvard Damm-Johnsen

6 Cohomology of curves

Contents

6.1 Constructible sheaves

Speaker: Håvard Damm-Johnsen

In this section, we assume as always that all schemes are locally Noetherian, and sheaves are assumed to be valued in Ab or $\mathrm{Mod}(\underline{\mathbb{Z}/n\mathbb{Z}})$, although most of the results extend to sheaves valued in modules over an arbitrary Noetherian ring.

Key motivation: we want a "nice" category for coefficient systems of schemes. Issue: the category of locally constant sheaves is not well-behaved, in particular, not closed under pushforward.

Example 6.1. Let G be a finite abelian group, and let $i\colon 0 \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$ be the inclusion of $0 = \operatorname{Spec}\mathbb{C}$ corresponding to the origin. Then $i_*\underline{G}$ is the skyscraper sheaf, and is not locally constant: if $U \to \mathbb{A}^1_{\mathbb{C}}$ is etale with 0 in its image, then the stalk of $i_*\underline{G}$ at 0 is different from the stalk away from 0 so i_*G not constant on any etale covering.

Recall that a sheaf $\mathscr{F}$ is *locally constant* on a scheme X if there exists some étale covering $\{\phi : U \to X\}$ such that $\mathscr{F}|_U := \phi^*\mathscr{F}$ is a constant sheaf.

We first define constructible sheaves on a Noetherian scheme:

Definition 6.2. Let X be a Noetherian scheme. A sheaf $\mathscr{F} \in \mathrm{Sh}(X_{\text{ét}})$ is **constructible** if there exists a finite partition $X = \bigsqcup_i Z_i$ where Z_i are locally closed subschemes of X, and $\mathscr{F}|_{Z_i}$ is locally constant with finite stalks.

We frequently shorten "locally constant with finite stalks" to "finite locally constant". The reason for restricting our attention to such sheaves is that cohomology with infinite coefficient sheaves is frequently ill-behaved, as in example 5.16.

Remark. In [Mil80, §V.1], Milne defines constructible sheaves via algebraic spaces, while we loosely follow the approach of [Sta21, Section 05BE], albeit in lesser generality.

Constructible sheaves extend the class of locally constant sheaves by allowing them to vary along closed subschemes. We can think of these as "locally locally constant sheaves", and by taking the trivial partition it is clear that any locally constant sheaf with finite stalks is constructible.

Lemma 6.3. *Let X be Noetherian. Then we can check constructibility Zariski-locally.*

Proof. We want to show that if $\mathscr{F}|_{U_i}$ is constructible for some Zariski-open covering $X = \bigcup_i U_i$, then $\mathscr{F}$ is constructible. Since X is quasi-compact, we can assume $\{U_i\}$ is finite. If $U_i = \sqcup_j Z_{ij}$ with $\mathscr{F}_{Z_{ij}}$ locally constant with finite stalks, then we have a decomposition $X = \bigcup_{ij} Z_{ij}$. By the usual topological argument, this can be refined to a disjoint union $X = \sqcup_{i'} Z'_{i'}$ with $\mathscr{F}|_{Z'_{i'}}$ finite locally constant. □

This allows us to extend the definition of constructibility to arbitrary locally Noetherian schemes in a natural way.

Proposition 6.4 ([Sta21, Tag 095H])**.** *Let $f : X' \to X$ be a finite étale morphism, and $\mathscr{F}' \in \mathrm{Sh}(X'_{\text{ét}})$ a constructible sheaf. Then $f_*\mathscr{F}'$ is also constructible.*

This is somewhat technical, and in the interest of time we won't go into details. The full subcategory of $\mathrm{Sh}(X_{\text{ét}})$ consisting of constructible sheaves retains several good properties:

Theorem 6.5. *The category of constructible sheaves is closed under closed under taking kernels, cokernels, extensions and tensor products, and is abelian.*

Tate twists

Tate twists are a nifty device for stating Poincaré duality without making a choice of an orientation. See this link for a less vague explanation.

Fix now a scheme X such that n is invertible in every residue field of X. Then we saw in exercise sheet 3 that μ_n is locally isomorphic to the constant sheaf $\underline{\mathbb{Z}/n\mathbb{Z}}$, and we regard μ_n as a locally free sheaf of $\underline{\mathbb{Z}/n\mathbb{Z}}$-modules of rank 1 on $X_{\text{ét}}$.

Lemma 6.6. *If $\mathscr{F}$ is a locally free and constructible sheaf with values in $\underline{\mathbb{Z}/n\mathbb{Z}}$, then so is its dual, $\mathscr{F}^\vee := \mathscr{H}om_{\underline{\mathbb{Z}/n\mathbb{Z}}}(\mathscr{F}, \underline{\mathbb{Z}/n\mathbb{Z}})$.*

Proof. Choose an étale covering $\{U \to X\}$ such that $\mathscr{F}|_U$ is free; then $\mathscr{F}^\vee|_U$ is also free. By lemma 6.3 we can check constructibility locally, where it is immediate. □

In particular, we can consider the dual of μ_n. Let

$$(\underline{\mathbb{Z}/n\mathbb{Z}})(r) := \begin{cases} \mu_n^{\otimes r} & \text{if} \quad r > 0, \\ \underline{\mathbb{Z}/n\mathbb{Z}} & \text{if} \quad r = 0, \\ \left(\mu_n^{\otimes(-r)}\right)^\vee & \text{if} \quad r < 0. \end{cases} \tag{6.1}$$

This is a sheaf of $\underline{\mathbb{Z}/n\mathbb{Z}}$-modules by the previous lemma.

Definition 6.7. Let $\mathscr{F}$ be a sheaf of $\underline{\mathbb{Z}/n\mathbb{Z}}$-modules, and fix $r \in \mathbb{Z}$. The **r-th Tate twist of $\mathscr{F}$** is $\mathscr{F}(r) := \mathscr{F} \otimes (\underline{\mathbb{Z}/n\mathbb{Z}})(r)$.

Proposition 6.8. *Let $\mathscr{F}$ be a constructible sheaf. Then $\mathscr{F}(r)$ is locally isomorphic to $\mathscr{F}$.*

6.2 Poincaré duality

A very readable introduction to this is Tony Feng's notes.

The intuition for Poincaré duality is most easily seen in the case of a real compact n-manifold M. Recall that for each for each $0 \leq k \leq n$, the *cup product*

$$H^k(M;\mathbb{R}) \times H^{n-k}(M;\mathbb{R}) \to H^n(M;\mathbb{Z}) \tag{6.2}$$

defines a non-degenerate bilinear map.

Theorem 6.9 (Classical Poincaré duality)**.** *Let M be an orientable compact manifold of real dimension n. Then a choice of an orientation on M defines a* trace map

$$\int_M : H^n(M;\mathbb{R}) \to \mathbb{R}, \tag{6.3}$$

which in turn gives an identification

$$H^i(M;\mathbb{R}) \cong \left(H^{n-i}(M;\mathbb{R})\right)^\vee \cong H_{n-i}(M;\mathbb{R}). \tag{6.4}$$

Since we don't have a canonical way of orienting our schemes, we ought to study the above when M is not oriented. In that case we need to add extra conditions on the coefficient ring; for example, we always have Poincaré duality for cohomology with coefficients in $\mathbb{Z}/2\mathbb{Z}$. This is done by introducing an *orientation sheaf*, whose analogue in the sheaf setting is $\mu^{\otimes r}$. Postponing some essential definitions, we give the statement of Poincare duality for algebraic curves:

Theorem 6.10 (Poincaré duality for curves, [Mil80, Thm. V.2.1])**.** *Let X be a smooth projective curve over an algebraically closed field k, and suppose $n \in \mathbb{Z}$ is invertible in k.*

(a) If $U \subset X$ is a non-empty open subscheme, then there is a canonical isomorphism

$$\eta(U) : H^2_c(U, \mu_n) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}. \tag{6.5}$$

(b) For any constructible sheaf $\mathscr{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $U_{\text{ét}}$, the groups $H^r_c(U, \mathscr{F})$ and $\operatorname{Ext}^r_{\operatorname{Sh}(U_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})}(\mathscr{F}, \mu_n)$ are finite for all r and vanish for $r > 2$. The pairing

$$H^r_c(U, \mathscr{F}) \times \operatorname{Ext}^r_{\operatorname{Sh}(U_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})}(\mathscr{F}, \mu_n) \to H^2_c(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z} \tag{6.6}$$

is non-degenerate.

Remark. The assumption that k be algebraically closed is necessary; however there are analogues for other fields, for example Tate-Poitou duality in Galois cohomology, and more generally Artin-Verdier duality for $\operatorname{Spec} \mathscr{O}_K$, when K is a number field. There are also many generalisations of Poincaré duality, in particular Verdier duality, see for example [KS13].

Speaker: Andrés Ibáñez Núñes

Let us first explain the cohomology groups H_c:

Definition 6.11. Let $j : U \hookrightarrow X$ be an open immersion, and $\mathscr{F} \in \operatorname{Sh}(U_{\text{ét}})$. Then $H^\bullet_c(U, \mathscr{F}) := H^\bullet(X, j_!\mathscr{F})$ is called **cohomology with compact support**.

We can define this more generally for $U \to \operatorname{Spec} k$ separated of finite type: by Nagata compactification ([Sta21, Theorem 0F41] or Brian Conrad's notes), it factors as

$$\begin{array}{ccc} U & \xrightarrow{j} & X \\ & \searrow & \downarrow \\ & & \operatorname{Spec} k \end{array} \tag{6.7}$$

where j is an open immersion and $X \to \operatorname{Spec} k$ is proper, and one can check that $H^\bullet_c(U, \mathscr{F})$ is independent of the choice of compactification X. One can also check that given a short exact sequence of sheaves on U, there is a corresponding long exact sequence in $H^\bullet_c$.

Next, let's define the pairing of eq. (6.6):[21] for transparency, fix an abelian category $\mathscr{A}$ with enough injectives, and let A, B and C be objects in $\mathscr{A}$. It is well-known (see e.g. Wikipedia) that $\operatorname{Ext}^r(A, B)$ classifies "r-extensions"

$$0 \to B \to X_r \to \ldots \to X_1 \to A \to 0 \tag{6.8}$$

[21] I found the explanation on Wikipedia a lot more intuitive than the one from the talk, so I decided to type up that.

up to equivalence, and if

$$\xi = 0 \to B \to X_r \to \ldots \to X_1 \to A \to 0 \quad \text{and} \quad \xi' = 0 \to C \to X'_s \to \ldots \to X'_1 \to B \to 0$$

are elements of $\mathrm{Ext}^r(A,B)$ and $\mathrm{Ext}^s(B,C)$ respectively, then there is a natural pairing

$$\begin{array}{ccc} \mathrm{Ext}^n(A,B) \times \mathrm{Ext}^m(B,C) & \to & \mathrm{Ext}^{n+m}(A,C) \\ (\xi,\xi') & \mapsto & \xi \smile \xi', \end{array} \tag{6.9}$$

where

$$\xi \smile \xi' = 0 \to C \to X'_s \to \ldots \to X'_1 \to X_s \to \ldots \to X_1 \to A \to 0, \tag{6.10}$$

the map $X'_1 \to X_r$ being the natural composition $X'_1 \to B \to X_r$. One then checks that $\xi \smile \xi'$ is actually an extension, hence a well-defined element of $\mathrm{Ext}^{r+s}(A,C)$.

Returning to our situation, we take $\mathscr{A} = \mathrm{Sh}(X_{\text{ét}}, \underline{\mathbb{Z}/n\mathbb{Z}})$, $A = \underline{\mathbb{Z}}$, $B = j_!\mathscr{F}$ and $C = j_!\mu_n$. Since Ext are the derived functors of Hom, we have $A = \mathrm{Ext}^r(\underline{\mathbb{Z}}, j_!\mathscr{F}) \cong H^r_c(U,\mathscr{F})$ and $C = \mathrm{Ext}^{r+s}(\underline{\mathbb{Z}}, j_!\mu)_n \cong H^{r+s}_c(U,\mu_n)$, and by the adjunction $(j^*, j_!)$ we find

$$B = \mathrm{Ext}^s(j_!\mathscr{F}, j_!\mu_n) \cong \mathrm{Ext}^s(\mathscr{F}, j^* j_!\mu_n) \cong \mathrm{Ext}^s(\mathscr{F},\mu_n). \tag{6.11}$$

Altogether, this gives the pairing in eq. (6.6). Now we are ready to sketch the main ideas of the proof of theorem 6.10.

Proof of a). We want to construct a canonical isomorphism $\eta(U)\colon H^2_c(U,\mu_n) \cong \mathbb{Z}/n\mathbb{Z}$. First assume $U = X$. Taking the long exact sequence in $H^\bullet_c = H^\bullet$ associated to the Kummer sequence (example 3.36), we get

$$\ldots \to H^1(X,\mathbb{G}_m) \xrightarrow{\cdot n} H^1(X,\mathbb{G}_m) \to H^2(X,\mu_n) \to H^2(X,\mathbb{G}_{m)} \to H^2(X,\mathbb{G}_m) \to \ldots \tag{6.12}$$

and we make a few observations:

- $H^m(X,\mathbb{G}_m) = 0$ for all $m \geq 2$ by Tsen's theorem.
- $H^1(X,\mathbb{G}_m) \cong \mathrm{Pic}(X)$ as in section 5.2.
- There is a natural short exact sequence $0 \to \mathrm{Jac}_k X \to \mathrm{Pic}(X) \to \mathbb{Z} \to 0$, where $\mathrm{Jac}_k(X)$ is the *Jacobian* of X.
- There is a surjective multiplication map $\mathrm{Jac}_k X \to \mathrm{Jac}_k X$.

Therefore we have a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Jac}_k(X) & \longrightarrow & \mathrm{Jac}_k(X) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H^1(X,\mathbb{G}_m) & \longrightarrow & H^1(X,\mathbb{G}_m) & \longrightarrow & H^2(X,\mu_n) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot n} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{6.13}
$$

and the snake lemma applied to the two bottom rows implies that the induced dashed map $H^2(X,\mu_n) \to \mathbb{Z}/n\mathbb{Z}$ is an isomorphism. For general $U \hookrightarrow X$, let $i\colon Z := X \setminus U \to Z$ be the natural inclusion, and recall that we have a short exact sequence

$$1 \to j_! j^* \mu_n \to \mu_n \to i_* i^* \mu_n \to 1, \tag{6.14}$$

inducing a long exact sequence

$$\ldots \to H^p(X, j_! j^* \mu_n) \cong H^p_c(U,\mu_n) \to H^p(X,\mu_n) \to H^p(X, i_* i^* \mu_n) \to H^{p+1}(X, j_! j^* \mu_n) \to \ldots . \tag{6.15}$$

We claim that $H^\bullet(X, i_* i^* \mu_n) = 0$: since $H^\bullet(X, i_* i^* \mu_n) = H^\bullet(Z, i^* \mu_n)$, and Z consists of a finite collection of points, $Z = \bigsqcup \operatorname{Spec} k$, Tsen's theorem implies that $H^\bullet(Z, i^* \mu_n) = 0$. Exactness of the long exact sequence and the proof for $U = X$ then gives $H^p_c(U,\mu_n) \cong H^p(X,\mu_n) \cong \mathbb{Z}/n\mathbb{Z}$ for all p. □

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