Étale cohomology reading seminar

Håvard Damm-Johnsen

5 First computations

Contents

5.1 Čech cohomology

Speaker: George Cooper

In this section we restrict our attention to étale sheaves, although several arguments remain true for general sites.

We can develop the machinery of Čech cohomology for the étale topology by analogy with the Zariski case, replacing $U \cap V$ with $U \times_X V$. However, some care must be taken; for example, $U \times_X U \neq U$ in general.

Fix a scheme X, an étale presheaf $\mathscr{F}$ on X, and let $\mathscr{U} = \{U_i \xrightarrow{\phi_i} X\}_{i \in \mathscr{I}}$ an étale cover of X. For $p > 0$ and $i_0, \dots, i_p \in \mathscr{I}$, set $U_{i_0,\dots,i_p} := U_{i_0} \times_X \dots \times_X U_{i_p}$. Then the natural projections onto the factors give rise to a map

$$\mathrm{pr}_{\hat{j}} \colon U_{i_0,\dots,i_p} \to U_{i_0,\dots,\hat{i}_j,\dots,i_p}, \qquad 0 \leq j \leq p, \tag{5.1}$$

where $\hat{i}_j$ means that the i_j-component is omitted. Next, let

$$\check{C}^p(\mathscr{U}, \mathscr{F}) := \prod_{(i_0,\dots,i_p)} \mathscr{F}(U_{i_0,\dots,i_p}) \tag{5.2}$$

and note that the projections $\mathrm{pr}_{\hat{j}}$ induce natural maps $\mathrm{res}_j = \mathscr{F}(\mathrm{pr}_{\hat{j}})$, which we use to define

$$d = d^p \colon \check{C}^p(\mathscr{U}, \mathscr{F}) \to \check{C}^{p+1}(\mathscr{U}, \mathscr{F}) \quad \text{by} \quad d^p(s_{i_0,\dots,i_p}) \sum_{j=0}^{p+1} (-1)^j \mathrm{res}_j(s_{i_0,\dots,\hat{i}_j,\dots,i_{p+1}}). \tag{5.3}$$

One then checks that $d^{p+1}d^p = 0$ for all $p \geq 0$, so $(\check{C}^\bullet(\mathcal{U},\mathcal{F}), d^\bullet)$ forms a complex, and we can take cohomology, giving $\check{H}^p(\mathcal{U},\mathcal{F}) := \ker d^{p+1} / \operatorname{im} d^p$. Then it follows directly from the definition that $\check{H}^0(\mathcal{U},\mathcal{F}) = \Gamma(X_{\text{ét}},\mathcal{F})$.

Note that $\check{C}^\bullet(\mathcal{U},\mathcal{F})$ depends on the choice of covering $\mathcal{U}$; to remove this dependency, we introduce the notion of a *refinement*:

Definition 5.1. A covering $\mathcal{V} = \{V_j \to X\}_{j\in\mathcal{J}}$ is a **refinement of** $\mathcal{U} = \{U_i \to X\}_{i\in\mathcal{I}}$ if there exists a map $\tau\colon \mathcal{J} \to \mathcal{I}$ such that for all $j \in \mathcal{J}$, there exists a map η_j such that triangle

$$\begin{tikzcd} V_j \arrow[rr, "\eta_j"] \arrow[dr] & & U_{\tau(j)} \arrow[dl] \\ & X & \end{tikzcd} \tag{5.4}$$

commutes.

Such a refinement gives maps

$$\tau^p : \check{C}^p(\mathcal{U},\mathcal{F}) \to \check{C}^p(\mathcal{V},\mathcal{F}), \qquad \tau^p(s)_{j_0,\ldots,j_p} = \operatorname{res}_{\eta_{j_0}\times\ldots\times\eta_{j_p}}(s_{\tau(j_0),\ldots,\tau_{j_p}}) \tag{5.5}$$

where $s = s_{i_0,\ldots,i_p} \in \check{C}^p(\mathcal{U},\mathcal{F})$. One then checks that $\tau d = d\tau$, so τ induces a map on cohomology, $\rho = \rho(\mathcal{V},\mathcal{U},\tau)$.

Lemma 5.2 ([Mil80, III.2.1])**.** *The map ρ does not depend on the choice of τ and η_j.*

Thus we can talk about a map $\rho = \rho(\mathcal{V},\mathcal{U})\colon \check{H}^\bullet(\mathcal{U},\mathcal{F}) \to \check{H}^\bullet(\mathcal{V},\mathcal{F})$. If $\mathcal{W}$ is a refinement of $\mathcal{V}$, then it is also a refinement of $\mathcal{U}$ (check!) and one can verify that $\rho(\mathcal{W},\mathcal{U}) = \rho(\mathcal{W},\mathcal{V})\rho(\mathcal{V},\mathcal{U})$.

Definition 5.3. The **p-th Čech cohomology group** of $(X,\mathcal{F})$ (for the étale topology) is given by

$$\check{H}^p(X_{\text{ét}},\mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^p(\mathcal{U},\mathcal{F}), \tag{5.6}$$

where the injective limit is taken over the poset of coverings $\mathcal{U}$ with maps ρ as above.

Remark. If $U \to X$ is an étale map and $\mathcal{F}$ a presheaf on the *big* étale site $X_{\text{Ét}}$, then the assignment

$$U \mapsto \check{H}^p(U,\mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^p(\mathcal{U},\mathcal{F}) \tag{5.7}$$

where $\mathcal{U}$ runs over coverings of U, is naturally a presheaf on $X_{\text{Ét}}$, denoted $\check{\mathcal{H}}^p(X_{\text{Ét}},\mathcal{F})$.

Proposition 5.4 ([Mil80, III.2.3-5])**.** *Fix $U \to X$ étale, and an étale covering $\mathcal{U}$ of U.*

(i) *For each* $p \geq 0$, $\check{H}^p(\mathcal{U}/U, -) : \mathrm{pSh}(X_{\text{ét}}) \to \mathrm{Ab}$ *is isomorphic to the* p*-th right derived functor of* $H^0(\mathcal{U}/U, -)$.

(ii) *For each* $p \geq 0$, $\check{H}^p(U, -) : \mathrm{pSh}(X_{\text{ét}}) \to \mathrm{Ab}$ *is isomorphic to the* p*-th right derived functor of* $H^0(U, -)$.

(iii) *For each* $p \geq 0$, $\check{H}^p(X_{\text{ét}}, -) : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Ab}$ *is isomorphic to the* p*-th right derived functor of* $\Gamma(X_{\text{ét}}, -)$ *if and only if for every* $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ *there exists a long exact sequence in Čech cohomology.*

Using spectral sequences, we can compute étale cohomology groups from Čech cohomology groups:

Proposition 5.5 ([Mil80, III.2.7])**.** *Let* $U \to X$ *be étale,* $\mathcal{U}$ *a covering of* U*, and* $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$*. Then we have spectral sequences starting on the second page as follows:*

(i) $\check{H}^p(\mathcal{U}/U, \mathcal{H}^q(\mathcal{F})) \Rightarrow \check{H}^{p+q}(U, \mathcal{F})$.

(ii) $\check{H}^p(\mathcal{U}, \mathcal{H}^q(\mathcal{F})) \Rightarrow \check{H}^{p+q}(U, \mathcal{F})$.

This essentially an application of the Grothendieck spectral sequence, see Milne for more details.

Corollary 5.6. *For any* $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ *and* $U \to X$ *étale, there are isomorphisms*

$$\check{H}^0(U, \mathcal{F}) \xrightarrow{\sim} H^0(U, \mathcal{F}) \quad \text{and} \quad \check{H}^1(U, \mathcal{F}) \xrightarrow{\sim} H^1(U, \mathcal{F}) \tag{5.8}$$

It is natural to ask whether we have isomorphisms in general; under mild conditions this is indeed the case:

Theorem 5.7 ([Mil80, Thm. III.2.17])**.** *Let* X *be a quasi-compact scheme, and suppose that any finite subset of* X *is contained in some affine open set.*[18] *Then for every* $p \geq 0$ *and* $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$*, we have natural isomorphisms*

$$\check{H}^p(X, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F}). \tag{5.9}$$

The proof is quite technical, and can be found in Milne's book. A further discussion on when derived cohomology and Čech cohomology differ can be found in the following link: MO.

5.2 Cohomology of the additive group scheme

Speaker: Mike Daas

In this section, the goal is to compute $H^1(X_{\text{ét}}, \mathbb{G}_m)$ when $X_{\text{ét}}$ is a sufficiently nice scheme of dimension 1. To do so, we first need to recall the language of divisors on schemes:

[18] For example, this includes projective schemes over an affine scheme.

Let X be a regular integral quasi-compact scheme with function field k, let $g\colon \operatorname{Spec} k \to X$ denote the structure morphism, and denote by $R(U)$ the collection of rational functions on U, for any $U \to X$ is étale. Recall from example 3.36 that $\mathbb{G}_{m,X} := \operatorname{Spec} \mathbb{Z}[t,t^{-1}] \times X$. Note that $R(U)^\times = \Gamma(U, g_* \mathbb{G}_{m,K})$, and the natural map $\Gamma(U, \mathcal{O}_U^\times) \to R(U)^\times$ induces an injection $r\colon \mathbb{G}_{m,K} \to g_* \mathbb{G}_{m,k}$.

Definition 5.8. The sheaf of **Cartier divisors** is the cokernel $\operatorname{Div} X := \operatorname{coker} r = g_* \mathbb{G}_{m,k} / r(\mathbb{G}_{m,k})$.

On the other hand, the notion of a Weil divisor extends naturally to schemes as follows: let X_1 denote the set of points of X of codimension 1. Then all the local rings $\mathcal{O}_{X,x}$ are discrete valuation rings, and we denote by $i_x : \{x\} \hookrightarrow X$ the natural inclusion of a point x into X.

Definition 5.9. The sheaf of **Weil divisors** is the sheaf $D_X := \bigoplus_{x \in X_1} i_{x*} \underline{\mathbb{Z}}$.

Under the conditions above, it is a standard result from scheme theory (eg. [Har77, Prop. II.6.11]) that $D_X \cong \operatorname{Div} X$, and we use the two interchangeably; for example, by definition of $\operatorname{Div} X$, D_X fits into an exact sequence

$$0 \to \mathbb{G}_{m,X} \to \mathbb{G}_{m,k} \to D_X \to 0. \tag{5.10}$$

Using the long exact sequence, we can therefore compute the cohomology of $\mathbb{G}_{m,X}$ in terms of the cohomology of $\mathbb{G}_{m,k}$ and D_X. For the latter, since cohomology commutes with direct sums, it suffices to determine the cohomology of $i_{x*}\underline{\mathbb{Z}}$ for all points $x \in X_1$. The Leray spectral sequence (theorem 4.23) for i_{x*} and $\underline{\mathbb{Z}}$ is

$$H^p(X_{\text{ét}}, R^q i_{x*} \underline{\mathbb{Z}}) \Rightarrow H^{p+q}(x, \underline{\mathbb{Z}}). \tag{5.11}$$

The right hand side is easier to compute explicitly. Let $\kappa(x)$ be the residue field at x and $G_x := \operatorname{Gal}(\kappa(x)^{\text{sep}}/\kappa(x))$. We claim that

$$H^i(x, \underline{\mathbb{Z}}) = \begin{cases} \Gamma(X, \underline{\mathbb{Z}}) = \mathbb{Z} & \text{for} \quad i = 0, \\ 0 & \text{for} \quad i = 1, \\ H^2(x, \mathbb{Z}) \hookrightarrow \operatorname{Hom}_{\text{cts}}(G_x, \mathbb{Q}/\mathbb{Z}) & \text{for} \quad i = 2. \end{cases} \tag{5.12}$$

Here the bottom line means that we can identify $H^2(x, \mathbb{Z})$ with a subgroup of $\operatorname{Hom}_{\text{cts}}(G_x, \mathbb{Q}/\mathbb{Z})$.

By using the equivalence $\operatorname{Sh}(\operatorname{Spec} \kappa(x)_{\text{ét}}) \cong \operatorname{Mod}(G_x)$, we can translate this to a computation in Galois cohomology. Indeed, we have that $H^1(x, \underline{\mathbb{Z}}) \subset \operatorname{Hom}_{\text{cts}}(G_x, \mathbb{Z})$ which vanishes since continuous homomorphisms factor through finite subgroups of $\mathbb{Z}$, and the only such is $\{0\}$; for $i = 2$, we use the long exact sequence arising from

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \tag{5.13}$$

along with the fact that $H^2(G_x, \mathbb{Q}) = 0$ since $\mathbb{Q}$ is an injective object, to show that $H^2(x, \underline{\mathbb{Z}}) \hookrightarrow \operatorname{Hom}_{\mathrm{cts}}(G_x, \mathbb{Q}/\mathbb{Z})$. An application of proposition 4.21 shows that $R^1 i_{x*}\underline{\mathbb{Z}} = 0$.

$$\begin{array}{ccc}
H^0(X_{\text{ét}}, R^2 i_{x*}\underline{\mathbb{Z}}) & H^1(X_{\text{ét}}, R^2 i_{x*}\underline{\mathbb{Z}}) & H^2(X_{\text{ét}}, R^2 i_{x*}\underline{\mathbb{Z}}) \\[2em]
H^0(X_{\text{ét}}, R^1 i_{x*}\underline{\mathbb{Z}}) = 0 & H^1(X_{\text{ét}}, R^1 i_{x*}\underline{\mathbb{Z}}) = 0 & H^2(X_{\text{ét}}, R^1 i_{x*}\underline{\mathbb{Z}}) = 0 \\[2em]
H^0(X_{\text{ét}}, i_{x*}\underline{\mathbb{Z}}) = \mathbb{Z} & H^1(X_{\text{ét}}, i_{x*}\underline{\mathbb{Z}}) & H^2(X_{\text{ét}}, i_{x*}\underline{\mathbb{Z}})
\end{array}$$

The E_2-page of the spectral sequence $H^p(X_{\text{ét}}, R^q i_{x*}\underline{\mathbb{Z}}) \Rightarrow H^{p+q}(x, \underline{\mathbb{Z}})$.

The spectral sequence[19] implies that

$$H^i(x, i_{x*}\underline{\mathbb{Z}}) = \begin{cases} \Gamma(X, i_{x*}\underline{\mathbb{Z}}) = \mathbb{Z} & \text{for} \quad i = 0, \\ 0 & \text{for} \quad i = 1, \\ H^2(x, i_{x*}\mathbb{Z}) \hookrightarrow \operatorname{Hom}_{\mathrm{cts}}(G_x, \mathbb{Q}/\mathbb{Z}) & \text{for} \quad i = 2. \end{cases} \tag{5.14}$$

and since cohomology commutes with direct sums,

$$H^i(X_{\text{ét}}, D_X) = \begin{cases} \bigoplus_{x \in X_1} \mathbb{Z} & \text{for} \quad i = 0, \\ 0 & \text{for} \quad i = 1, \\ H^2(X_{\text{ét}}, D_X) \hookrightarrow \bigoplus_{x \in X_1} \operatorname{Hom}_{\mathrm{cts}}(G_x, \mathbb{Q}/\mathbb{Z}) & \text{for} \quad i = 2. \end{cases} \tag{5.15}$$

On the other hand, the Leray spectral sequence associated to $g \colon \operatorname{Spec} k \to X$ and $\mathbb{G}_m$ is given by

$$H^p(X_{\text{ét}}, R^q g_* \mathbb{G}_{m,K}) \Rightarrow H^{p+q}(\operatorname{Spec} k, \mathbb{G}_m). \tag{5.16}$$

Recall from theorem 4.22 that the stalk of $R^q g_* \mathbb{G}_{m,k}$ at a geometric point $\overline{x}$ is given by $H^q(\operatorname{Frac} \mathcal{O}_{X,\overline{x}}, \mathbb{G}_m)$. By Hilbert's theorem 90, we know that $H^1(\operatorname{Frac} O_{X,\overline{x}}, \mathbb{G}_m) = 0$, so $R^1 g_* \mathbb{G}_{m,K} = 0$. By a similar argument as for $\underline{\mathbb{Z}}$, by passing to Galois cohomology we find that

$$H^0(X_{\text{ét}}, g_* \mathbb{G}_{m,k}) = H^0(k, \mathbb{G}_m) = k^\times \quad \text{and} \quad H^0(X_{\text{ét}}, g_* \mathbb{G}_{m,k}) \hookrightarrow H^2(k, \mathbb{G}_{m,k}). \tag{5.17}$$

[19] Explicitly, one argues as follows: In degree 1, we know one of the two terms vanishes, and the other one does not admit nontrivial boundary maps. The result in degree 1 must be zero, so both terms on the E2-page in degree 1 must be zero.

In degree 2, we just include one of the objects in the sequence in degree 2 into the total complex: again no boundary maps reach the term we are looking at and the total complex can be recovered from the degree 2 terms by some kind of filtration.

The long exact sequence associated to eq. (5.10) then becomes

$$\begin{aligned} 0 \to \Gamma(X_{\text{ét}}, \mathcal{O}_X^\times) \to k^\times \to \bigoplus_{x \in X_1} i_{x*}\mathbb{Z} \to H^1(X_{\text{ét}}, \mathbb{G}_m) \to 0 \\ 0 \to H^2(X, \mathbb{G}_m) \to H^2(k, \mathbb{G}_{m,k}). \end{aligned} \tag{5.18}$$

In particular, we can identify $H^1(X_{\text{ét}}, \mathbb{G}_m)$ with the quotient $\mathrm{Div}_X / (k^\times / \mathcal{O}_X^\times) = \mathrm{Pic}(X)$.

Stronger results:

Suppose now $\dim X = 1$, and that $\kappa(x)$ is perfect for every $x \in X$. Then $\mathcal{O}_{X,\overline{x}}$ is a Henselian discrete valuation ring with algebraically closed residue field, and we set $k_{\overline{x}} \coloneqq \mathrm{Frac}\, \mathcal{O}_{X,\overline{x}}$. We claim that $H^2(k_{\overline{x}}, \mathbb{G}_m) = 0$. Indeed, by [Ser02, II.2.2] we can identify $H^2(k_{\overline{x}}, \mathbb{G}_m)$ with the *Brauer group* $\mathrm{Br}(k_{\overline{x}})$ of $k_{\overline{x}}$, that is, the group of $k_{\overline{x}}$-algebras A with centre equal to $k_{\overline{x}}$, and whose only two-sided ideals are the trivial ones. The group operation is given by the tensor product, $- \otimes_{k_{\overline{x}}} -$. The valuation on $k_{\overline{x}}$ extends uniquely (by Henselian-ness) to a valuation on any $A \in \mathrm{Br}(k_{\overline{x}})$, and we can then produce a subfield $L' \subset A$ with $[A : k_{\overline{x}}] = [L' : k_{\overline{x}}]^2$. $L'/k_{\overline{x}}$ is unramified, so $L' = k_{\overline{x}}$ hence $A = k_{\overline{x}}$. See [Ser95, §XII.2] for more details.

Now, since the stalk of $R^2 g_* \mathbb{G}_{m,k}$ at $\overline{x}$ is given by $H^2(k_{\overline{x}}, \mathbb{G}_m)$, and all stalks vanish, we conclude that $R^2 g_* \mathbb{G}_{m,k} = 0$. The points $x \in X_1$ are all closed, so the functors i_{x*} are all exact by proposition 3.51.

It follows that $H^2(X_{\text{ét}}, g_* \mathbb{G}_{m,k}) = H^2(\operatorname{Spec} k, \mathbb{G}_m)$, and as before we get isomorphisms $H^q(X_{\text{ét}}, i_{x*}\mathbb{Z}) \cong H^q(x, \mathbb{Z})$, whence we obtain an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^2(X_{\text{ét}}, \mathbb{G}_m) \arrow[r] & H^2(k, \mathbb{G}_{m,k}) \arrow[r] & \bigoplus_{x \in X_1} \mathrm{Hom}_{\mathrm{cts}}(G_x, \mathbb{Q}/\mathbb{Z}) \arrow[dll, out=0, in=180] \\ & H^3(X_{\text{ét}}, \mathbb{G}_m) \arrow[r] & H^3(k, \mathbb{G}_{m,k}) & \end{tikzcd} \tag{5.19}$$

If we additionally assume that X is "excellent" (a technical condition we won't define, see [Sta21, Section 07QS] – a scheme is excellent if it can be covered by spectra of excellent rings), then $k_{\overline{x}}$ is quasi-algebraically closed field: any polynomial over it whose number of variables is greater than its degree has a root.

Example 5.10. The Chevalley–Warning theorem ([Ser73, Thm. I.3]) states that any finite field $\mathbb{F}_q$ is quasi-algebraically closed.

With X as above, for any closed point $x \in X$ we have $H^q(k_{\overline{x}}, \mathbb{G}_m) = 0$ hence

$R^q g_* \mathbb{G}_m = 0$ for $q > 0$, and this gives the long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H^r(X_{\text{ét}}, \mathbb{G}_m) \arrow[r] & H^r(k, \mathbb{G}_{m,k}) \arrow[r] & \bigoplus_{x \in X_1} H^{r-1}(G_x, \mathbb{Q}/\mathbb{Z}) \arrow[dlll, out=0, in=180] \\ H^{r+1}(X_{\text{ét}}, \mathbb{G}_m) \arrow[r] & H^{r+1}(k, \mathbb{G}_{m,k}) \arrow[rr] & & \cdots \end{tikzcd} \tag{5.20}$$

For example, if X is a smooth algebraic curve over an algebraically closed field, then k is C_1 by Tsen's theorem ([Sta21, Theorem 03RD]). Since we also have that $H^r(G_x, \mathbb{Q}/\mathbb{Z}) = 0$ for $r \geq 1$, as $k_{\overline{x}}$ is algebraically closed, we deduce from the long exact sequence that $H^r(X_{\text{ét}}, \mathbb{G}_m) = 0$ for all $r \geq 2$.

5.3 Comparing topologies

In this section, we take a brief pause from the gritty computations to answer the question "how do we compare cohomology groups defined with respect to different Grothendieck topologies on a site?" We don't give any proofs, but refer the eager reader to Milne's book.

Proposition 5.11 ([Mil80, Prop. III.3.1]). *Let $(C'/X)_E$ be a Grothendieck topology, let $C \subset C'$ be a subcategory and suppose $f\colon (C'/X)_E \to (C/X)_E$ is the map induced by the inclusion functor $C \hookrightarrow C'$. For any sheaves $\mathcal{F}'$ on $(C'/X)_E$ and $\mathcal{F}$ on $(C/X)_E$, we have*

$$H^i(X, f_*\mathcal{F}') \cong H^i(X, \mathcal{F}') \quad \text{and} \quad H^i(X, \mathcal{F}) \cong H^i(X, f^*\mathcal{F}'), \tag{5.21}$$

for all $i \geq 0$.

In particular, we can pass freely between the small and big étale sites when computing étale cohomology groups.

Definition 5.12. Let $(C_1/X)_{E_1}$ and $(C_2/X)_{E_2}$ be sites where $C_1 \supset C_2$ and $E_1 \supset E_2$. If for every covering in the E_1-topology there exists a covering in E_2 which refines it and vice versa, then we say that E_1 and E_2 admit **mutual refinements**.

Proposition 5.13 ([Mil80, Prop. III.3.3]). *Suppose E_1 and E_2 as above are* stable classes, *cf. definition 2.4. Let $f\colon (C_1/X)_{E_1} \to (C_2/X)_{E_2}$ be the natural map. If E_1 and E_2 admit mutual refinements, then*

$$H^i(X_{E_2}, f_*\mathcal{F}) \cong H^i(X_{E_1}, \mathcal{F}), \tag{5.22}$$

for any sheaf $\mathcal{F} \in \mathrm{Sh}(X_{E_1})$ and $i \geq 0$.

We can use this to restrict from the class of étale morphisms to the class of étale morphisms of finite type, or to the class of separated étale morphisms, or even to affine étale morphisms (exercise! – this amounts to showing that suitable mutual refinements exist). In a similar fashion we can reduce a problem from the class of smooth morphisms to the class of étale morphisms: the key point in showing this is that every smooth morphism admits a section étale-locally. The following shows that we can also restrict our attention to finite subcoverings:

Proposition 5.14 ([Mil80, Prop. III.3.5])**.** *Suppose $(C/X)_E$ is a* Noetherian site, *meaning that every covering has a finite subcovering, i.e. a covering consisting of finitely many elements. Let E_f denote the category of finite subcoverings. Then the categories of sheaves (resp. presheaves) on X_E and X_{E_f} are canonically equivalent. In particular, cohomology is preserved when passing from one to the other.*

5.4 Étale and complex cohomology

In this section, for a scheme X over $\operatorname{Spec}\mathbb{C}$, we let $H^i(X(\mathbb{C}),-)$ denote the usual singular cohomology. If étale cohomology is indeed a "good" cohomology theory, then it should coincide with singular cohomology under suitably nice conditions. The goal of this section is to prove the following theorem:

Theorem 5.15. *Let X be a smooth scheme over* $\operatorname{Spec}\mathbb{C}$ *and M a finite abelian group. Then*

$$H^i(X(\mathbb{C}),M) \cong H^i(X_{\text{ét}},\underline{M}), \tag{5.23}$$

for all $i \geq 0$.

Example 5.16. Note that it is crucial to assume M is finite; for example, if X is an elliptic curve, we have

$$H^1(X(\mathbb{C}),\mathbb{Z}) = \mathbb{Z}^2 \quad \text{but} \quad H^1(X_{\text{ét}},\mathbb{Z}) = \operatorname{Hom}_{\text{cts}}(\pi_1(X),\mathbb{Z}) = 0, \tag{5.24}$$

the latter because $\pi_1(X)$ is a profinite group, by the same argument as in section 5.2.

Proof (sketch). For $i = 0$, this amounts to showing that the numbers of components of X and $X(\mathbb{C})$ agree. This follows from a reduction to the case of X being a projective curve, and then an appeal to Riemann-Roch. For details, see [Sha13, §VII.2].

For $i = 1$, we use the fact that $H^1(X_{\text{ét}},\underline{M})$ is in bijective correspondence with the set of Galois coverings with automorphism group equal to M; see [**?** , Props. 11.1 & 11.3]. On the other hand, since $H^1(X(\mathbb{C}),M)$ classifies analytic covering spaces with automorphism group M, the case follows from the following theorem:

Theorem 5.17 (Riemann existence theorem)**.** *Let $X \to \operatorname{Spec}\mathbb{C}$ be locally of finite type. Then there is an equivalence of categories*

$$\{\text{finite étale covers } Y \to X\} \leftrightarrow \{\text{analytic covering spaces } Y^{\mathrm{an}} \to X^{\mathrm{an}}\},$$
$$Y \mapsto Y^{\mathrm{an}}. \tag{5.25}$$

Finally, for $i > 1$, let X_{cx} be the site on X^{an} where coverings are given by local isomorphisms of complex analytic spaces. Since for any complex-open $U \subset X(\mathbb{C})$ the map $U \hookrightarrow X(\mathbb{C})$ is a local isomorphism, we have a natural map $X_{\mathrm{cx}} \to X(\mathbb{C})_{\mathrm{top}}$, where $X(\mathbb{C})_{\mathrm{top}}$ denotes the site generated by the complex topology of $X(\mathbb{C})$. It is not difficult to see that these admit mutual refinements, so by proposition 5.14,

$$H^i(X_{\mathrm{cx}}, \underline{M}) \cong H^i(X_{\mathrm{top}}, \underline{M}). \tag{5.26}$$

By the implicit function theorem (cf. exercise sheet 1), for $U \to X$ étale that associated map $U^{\mathrm{an}} \to X^{\mathrm{an}}$ is a local isomorphism, giving rise to a map of sites $f\colon X_{\mathrm{cx}} \to X_{\text{ét}}$. This gives rise to a Leray spectral sequence

$$H^i(X_{\text{ét}}, R^j f_* \mathscr{F}) \Rightarrow H^{i+j}(X_{\mathrm{cx}}, \mathscr{F}). \tag{5.27}$$

If we can show that $R^j f_* \mathscr{F} = 0$ for $j > 0$, then the spectral sequence degenerates and we are done. By proposition 4.21, $R^j f_* \mathscr{F}$ is the sheafification of $U \mapsto H^j(U_{\mathrm{cx}}, \mathscr{F})$. The final ingredient in the proof is the following lemma, which relies on quite heavy machinery, namely [Mil80, VI.4.2 & 5.1]. We refer the reader to Milne's book for a proof.

Lemma 5.18 ([Mil80, III. 3.15])**.** *For a locally constant sheaf $\mathscr{F} \in \mathrm{Sh}(X_{\mathrm{cx}})$ with finite fibres and $i > 0$, fix $\gamma \in H^i(X_{\mathrm{cx}}, \mathscr{F})$. For any $x \in X(\mathbb{C})$, there exists an étale morphism $U \to X$ whose image contains x with $\gamma|_{U_{\mathrm{cx}}} = 0$.*

In particular, for our constant sheaf $\underline{M}$, we have that $H^i(U_{\mathrm{cm}}, \underline{M}) = 0$ and upon sheafifying this gives that $R^j f_* \mathscr{F} = 0$, as required. ☐

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