Étale cohomology reading seminar

Håvard Damm-Johnsen

4 Cohomology

Contents

4.1 Cohomology on sites

Speaker: Lukas Kofler

We assume some familiarity with the cohomological machinery used in algebraic geometry, and only give a quick summary here to fix notations. Further details can be found in [Har77, Chap. III].

Let $\mathscr{A}$ be an abelian category, and recall that an object I of $\mathscr{A}$ is *injective* if the functor $\mathrm{Hom}(-, I)$ is exact. We say $\mathscr{A}$ *has enough injectives* if for every element of $\mathscr{A}$ there exists an injection $A \hookrightarrow I$ where I is injective.

Proposition 4.1 ([Mil80, Prop. III.1.1])**.** *The category of sheaves valued in abelian groups on a site has enough injectives.*

In any abelian category $\mathscr{A}$ along with a left exact functor $F\colon \mathscr{A} \to \mathscr{A}'$, we can form right derived functors R^iF, $i \geq 0$, in the usual manner. These are characterised by the properties $R^0F = F$, $R^iF(I) = 0$ for any injective object I, and that every short sequence in $\mathscr{A}$ gives a long exact sequence in cohomology in $\mathscr{A}'$.

Example 4.2. The *global sections functor* $\Gamma(X, -)\colon \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Ab}$ is left exact, and we define $H^i(X, -) := R^i\Gamma(X, -)$ to be the corresponding cohomology functors.

Example 4.3. The inclusion $\mathrm{Sh}(X_{\text{ét}}) \hookrightarrow \mathrm{pSh}(X_{\text{ét}})$ is left exact by proposition 3.40 (i), and the cohomology functors are denoted by $\underline{H}^i(-)$.

Example 4.4. For a fixed sheaf $\mathscr{F} \in \mathrm{Sh}(X_{\text{ét}})$, the functor $\mathrm{Hom}(-, \mathscr{F})$ is left exact, and the right derived functors are denoted by $\mathrm{Ext}^i_{\mathrm{Sh}(X_{\text{ét}})}(\mathscr{F}, -)$.

Example 4.5. Similarly, for $\mathscr{F},\mathscr{G} \in \mathrm{Sh}(X_{\text{ét}})$ we can define the hom-sheaf by

$$\mathscr{H}\!om(\mathscr{F},\mathscr{G})\colon U \mapsto \mathrm{Hom}(\mathscr{F}|_U,\mathscr{G}|_U). \tag{4.1}$$

This gives a left exact functor $\mathscr{H}\!om(\mathscr{F},-)\colon \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$, with right derived functors $\mathscr{E}\!xt^i(\mathscr{F},-)$.

Example 4.6. For a continuous map of sites $\pi\colon X'_{E'} \to X_E$, the pushforward π_* is left exact, and the right derived functors $R^i\pi_*$ are called *higher direct images*.

4.2 Spectral sequences

Spectral sequences have a reputation for being somewhat arcane objects, and so we begin the section gently with some motivation:

Suppose we have a double complex $\{E_0^{p,q}\}_{p,q\geq 0}$ in an abelian category $\mathscr{A}$; that is, a collection of objects $E_0^{0,0}, E_0^{1,0}, E_0^{0,1}, \ldots$ along with maps

$$d_h\colon E_0^{p,q} \to E_0^{p+1,q} \quad \text{and} \quad d_v\colon E_0^{p,q} \to E_0^{p,q+1} \tag{4.2}$$

satisfying $d_h^2 = 0 = d_v^2$ and $d_h d_v = -d_v d_h$. These arise naturally in algebraic geometry, say from taking resolutions of complex, or complexes of filtered objects. From this double complex we construct the *total complex* $E_0^\bullet$ with $E_0^k := \bigoplus_i E^{i,k-i}$, the direct sum along the k-th antidiagonal. This is becomes a complex with the differential $d := d_h + d_v$.

It is natural to ask whether one can find the cohomology of the total complex by computing cohomology of the complexes in the horisontal or vertical directions separately. Taking cohomology of $E_0^{\bullet,\bullet}$ first in the vertical direction under the action of d_v, we define

$$E_1^{p,q} := \frac{\ker d_v^{p,q}}{\operatorname{im} d_v^{p,q-1}}. \tag{4.3}$$

This gives a new double complex, where the action of the induced maps d_v is trivial. However, the induced maps $d_h\colon E_1^{p,q} \to E_1^{p+1,q}$ are well-defined (check!) and non-zero in general. By convention they are denoted d_1, and $E_1^{\bullet,\bullet}$ is called *the first page*. We can now take its cohomology under d_1, and the resulting double complex is denoted by $E_2^{\bullet,\bullet}$, called *the second page*.

Now, one might think that we are done at this point, and should be able to say something about the cohomology of the total complex. In fact, if the only non-zero columns of $E_2^{\bullet,\bullet}$ are given by p and $p+1$, then we have an exact sequence

$$0 \to E_2^{p,q} \to H^{p+q}(E^\bullet) \to E_2^{p+1,q-1} \to 0, \tag{4.4}$$

so we have computed $H^n(E^\bullet)$ "up to extension".

However, in general there is a new non-zero differential on $E_2^{\bullet,\bullet}$, $d_2 \colon E_2^{p,q} \to E_2^{p+2,q-1}$ constructed by the following diagram chase:

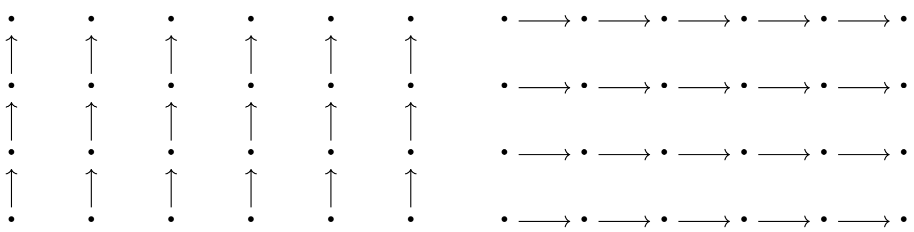
Take $x \in E_2^{p,q}$ and lift it to $x' \in E_1^{p,q}$. Then $d_1(x') = 0$ in $E_1^{p+1,q}$, so for a lift $x'' \in E_0^{p,q}$ of x', $d_h(x'')$ is in the image of d_v, say $d_h(x'') = d_v(y)$ for $y \in E_0^{p+1,q-1}$. Now $d_h(y) \in E_0^{p+2,q-1}$ and $d_v d_h(y) = -d_h d_v(y) = -d_h^2(x'') = 0$, so $d_h(y) \in \ker d_v$, determining an element of $E_1^{p+2,q-1}$. Since $d_1 d_h)(y) = 0$, this factors through to an element of $E_2^{p+2,q-1}$, which is the desired image of x.

Now that we have defined the map, it is not too difficult to check that it is well-defined and a differential, and in fact this construction generalises to higher differentials $d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$. This leads to the following definition:

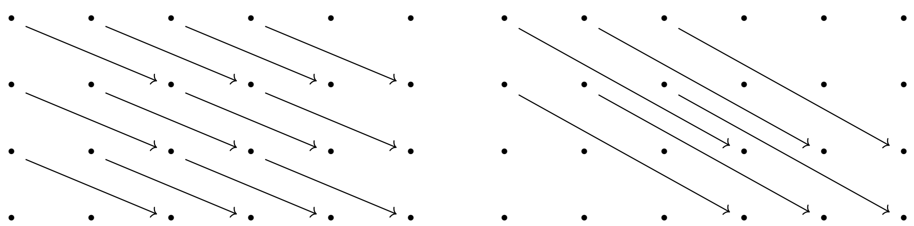
Definition 4.7. A (cohomological, first quadrant) **spectral sequence** consists of

(i) objects $E_r^{p,q} \in \mathscr{A}$ for all $p, q, r \geq 0$,
(ii) morphisms $d_r \equiv d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ satisfying $d_r^2 = 0$,
(iii) isomorphisms $\ker d_r^{p,q} / \operatorname{im} d_r^{p-r,q-r+1} \cong E_{r+1}^{p,q}$.

The collection $\{E_r^{p,q}\}_{p,q \geq 0}$ is called **the r-th page** of the spectral sequence.



E_0 and E_1.



E_2 and E_3.

Note that since $E_r^{p,q}$ all lie in the upper quadrant, we eventually take the cohomology of $0 \to E_r^{p,q} \to 0$, the zeroes lying in the second and fourth quadrant respectively. When this happens, we evidently have $E_r^{p,q} = E_{r+1}^{p,q} = \dots$, and we write $E_r^{p,q} = E_\infty^{p,q}$.

Theorem 4.8. *For each $n \geq 0$, there is a decreasing filtration on $H^n(E^\bullet)$,*

$$H^n = F^0 H^n \supset \ldots \supset F^{n+1} H^n = 0, \tag{4.5}$$

such that $\mathrm{gr}_p H^n = E_\infty^{p,n-p}$.[14]

In particular, we have that $\bigoplus_{p=0}^n E_\infty^{p,n-p} = \mathrm{gr}\, H^n$. We write $E_0^{p,q} \Rightarrow H^{p+q}(E^\bullet)$ and say that the spectral sequence **converges to** $H^{p+q}(E^\bullet)$.

Note that we have not quite computed the cohomology of the total complex, but if for some $r \geq 2$ we have that $E_r^{\bullet,\bullet}$ has only one non-zero column or row, then we can read off $H^n(E^\bullet)$ directly. In this case we say that the spectral sequence **collapses, or degenerates, at page** r. In most applications, spectral sequences already collapse at E_1 or E_2.

A powerful feature of spectral sequences is that we can flip the roles of d_h and d_v while still converging to the cohomology of the graded complex. For convenience, let $\hat{E}$ denote the original spectral sequence and $\vec{E}$ the one with d_h and d_v swapped, and let's look at some applications:

Example 4.9 (Five lemma)**.** Suppose we have the following diagram

$$\begin{tikzcd}
F \arrow[r] & G \arrow[r] & H \arrow[r] & I \arrow[r] & J \\
A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & D \arrow[r] \arrow[u, "\delta"] & E \arrow[u, "\epsilon"]
\end{tikzcd} \tag{4.6}$$

where the rows are exact, and α, β, δ and ϵ are isomorphisms. The *five lemma* states that in this case γ is an isomorphism as well. We can show this using a spectral sequence argument: view the diagram in eq. (4.6) as $\vec{E}_0^{p,q}$, and take horisontal cohomology to get $\vec{E}_1^{p,q}$, which since the rows are exact looks as follows:

$$\begin{tikzcd}
? & 0 & 0 & 0 & ? \\
? \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & ? \arrow[u]
\end{tikzcd} \tag{4.7}$$

Now the cohomology of the total complex vanishes in the degrees corresponding to $H \to C$. The spectral sequence converges at the 2nd page since there are no more arrows between non-zero objects to draw there.

Now let's look at the vertical cohomology. The first page, $\hat{E}_1^{p,q}$, looks as follows:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd} \tag{4.8}$$

[14]Recall that the *p-th graded part* of the filtered object $F^\bullet H^n$ is given by the quotient $F^{p+1} H^n / F^p H^n$, which is the p-th summand of $\mathrm{gr}\, H^n := \bigoplus_{k \geq 0} F^{k+1} H^n / F^k H^n$.

and γ being an isomorphism is equivalent to the vanishing of the two question marks here. Note that the spectral sequence converges on this page, and so since the question marks correspond to the same pieces of the cohomology which vanished by the previous computation, we conclude that γ is indeed an isomorphism. This proves the claim.

Example 4.10 (Long exact sequence in cohomology)**.** Using spectral sequences we can also deduce the long exact sequence in cohomology from a short exact sequence of objects. Suppose

$$0 \to A \to B \to C \to 0 \tag{4.9}$$

is an exact sequence. In horisontal cohomology the sequence converges on the first page because the sequence is exact. On the other hand, $\hat{E}_1^{p,q}$ is given by

$$\begin{tikzcd}
0 \arrow[r] & H^2(A) \arrow[r, "\alpha_2"] & H^2(B) \arrow[r, "\beta_2"] & H^2(C) \arrow[r] & 0 \\
0 \arrow[r] & H^1(A) \arrow[r, "\alpha_1"] & H^1(B) \arrow[r, "\beta_1"] & H^1(C) \arrow[r] & 0 \\
0 \arrow[r] & H^0(A) \arrow[r, "\alpha_0"] & H^0(B) \arrow[r, "\beta_0"] & H^0(C) \arrow[r] & 0
\end{tikzcd} \tag{4.10}$$

and the next page looks like this:

$$\begin{tikzcd}
0 & \ker\alpha_2 \arrow[drr] & \ker\beta_2/\operatorname{im}\alpha_2 & \operatorname{coker}\beta_2 & 0 \\
0 & \ker\alpha_1 \arrow[drr] & \ker\beta_1/\operatorname{im}\alpha_1 & \operatorname{coker}\beta_1 & 0 \\
0 & \ker\alpha_0 & \ker\beta_0/\operatorname{im}\alpha_0 & \operatorname{coker}\beta_0 & 0
\end{tikzcd} \tag{4.11}$$

The sequence converges on the next page, and so we conclude that every entry on the next page is 0. In particular, $\ker\beta_i/\operatorname{im}\alpha_i = 0$ and $\ker\alpha_{i+1} \cong \operatorname{coker}\beta_i$ for all $i \geq 0$. This gives the connecting homomorphism $H^i(C) \to H^{i+1}(A)$ along with exactness everywhere in the long exact sequence.

Exercise. Prove the snake lemma using spectral sequences: given a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] & F \arrow[r] & G \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd} \tag{4.12}$$

prove the exactness of the sequence

$$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \to \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma \to 0. \tag{4.13}$$

Next we turn to study properties of spectral sequences. For convenience, we consider only sequences $E = \hat{E}$, with d_0 vertical. We will also abstract slightly and let $E^{\bullet,\bullet}$ converge to any family of filtered objects $E^n \in \mathscr{A}$ such that $F^0 E^n = E^n$ and $F^{n+1} E^n = 0$. As usual we then have $E_\infty^{p,q} \cong \mathrm{gr}_p E^{p+q}$.

Note that $E_{r+1}^{p,q}$ is a subquotient of $E_r^{p,q}$ for all p, q, r; this gives rise to a sequence of quotient maps

$$E_0^{n,0} \to E_1^{n,0} \to \dots \to E_\infty^{n,0}. \tag{4.14}$$

The natural composite $E_0^{n,0} \to E^n$ is called an *edge morphism*. In a similar manner we construct an edge morphism $E^n \to E_0^{0,n}$.

Exercise. Show that the following sequence is exact:

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \xrightarrow{d} E_2^{2,0} \to E^2. \tag{4.15}$$

This is called the **five term exact sequence**.

Example 4.11. The Hochschild-Serre spectral sequence in group cohomology computes the group cohomology of a group G in terms of a subgroup H and the quotient G/H. In this case, the five term exact sequence is simply the inflation-restriction sequence

$$0 \to H^1(G/H, A^H) \to H^1(G, A) \to H^1(H, A)^{G/H} \to H^2(G/H, A^H) \to H^2(G, A). \tag{4.16}$$

We round off the section with a theorem, the "chain rule for derived functors", which we will put to good use later:

Theorem 4.12 (The Grothendieck spectral sequence)**.** *Let* A, B *and* C *be abelian categories, with* A *and* B *having enough injectives. Suppose we are given left exact functors* $\mathrm{A} \xrightarrow{G} \mathrm{B} \xrightarrow{F} \mathrm{C}$ *such that for any injective object* $I \in \mathrm{A}$, $R^i F(I) = 0$ *for* $i > 0$*. Then there exists a convergent (first quadrant, cohomological) spectral sequence starting on the page* 2*:*

$$E_2^{p,q} = (R^p F)(R^q G)(A) \Rightarrow R^{p+q}(FG)(A). \tag{4.17}$$

For a proof of this, see [Wei94, Sec. 5.8].

4.3 Étale cohomology groups

Speaker: George Robinson

Étale cohomology and Galois cohomology

Let k be a field, $X = \operatorname{Spec} k$ and for the remainder of the section, $G \coloneqq \operatorname{Gal}(k^{\mathrm{sep}}/k)$ for some fixed separable closure k^{sep} of k.

Recall from theorem 3.12 that $\operatorname{Sh}(X_{\text{ét}})$ is equivalent to $\operatorname{Mod}(G)$, the category of discrete G-modules. Explicitly, for $M \in \operatorname{Mod}(G)$ we have a sheaf $\mathscr{F}_M$ whose sections over a finite separable extension k'/k are given by $M^{G'}$, the elements of M fixed by $G' = \operatorname{Gal}(k^{\mathrm{sep}}/k')$. In the equivalence, the functor $\Gamma(X,-)$ simply becomes the covariant functor $(-)^G \colon \operatorname{Mod}(G) \to \mathrm{Ab}$, which sends a G-module M to the G-invariant submodule M^G. Taking derived functors shows that $H^{\bullet}_{\text{ét}}(X,\mathscr{F}_M) = H^{\bullet}_{\mathrm{Gal}}(K,M)$, that is, the derived functors of $\Gamma(X,-)$ in the étale topology are precisely Galois cohomology.

Example 4.13. With $X = \operatorname{Spec} k$ as above, suppose M is a trivial G-module, that is, M is an abelian group with the trivial action of G, $g \cdot m = m$ for all $g \in G$ and $m \in M$. Then $H^0(G,M) = M$, and from the definition of a cocycle we see that $H^1(G,M) = \operatorname{Hom}(G,M)$.

This is already nontrivial, as for example

$$\begin{aligned} H^1(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),\mathbb{Z}/2\mathbb{Z}) &= \operatorname{Hom}(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),\mathbb{Z}/2\mathbb{Z}) \\ &\cong \{\text{extensions of degree dividing } 2\} \cong \mathbb{Q}^\times/(\mathbb{Q}^\times)^2. \end{aligned} \tag{4.18}$$

An important theorem whose geometric analogue we will encounter later, is the following:

Theorem 4.14 (Hilbert's theorem 90)**.** *If k is a perfect field, then $H^1(G,\overline{k}^\times) = 0$.*

We can apply this to the Kummer sequence (example 3.36) to compute the cohomology of μ_n regarded as a Galois module:

Example 4.15. Applying Galois cohomology to the sequence

$$0 \to \mu_n(\overline{k}) \to \overline{k}^\times \xrightarrow{x \mapsto x^n} \overline{k}^\times \to 0 \tag{4.19}$$

gives the following (rather short) long exact sequence in cohomology,

$$0 \to \mu_n(\overline{k})^G = \mu_n(k) \to k^\times \to k^\times \to H^1(G,\mu_n) \to H^1(G,\overline{k}^\times) = 0, \tag{4.20}$$

where the last equality is Hilbert's theorem 90. We sometimes write this as $H^1(G,\mu_n) = k^\times/(k^\times)^n$. Note that this generalises the previous example, since for $k = \mathbb{Q}$, G acts trivially on $\mu_2 = \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$.

Taking limits in two different ways gives the two identities

$$H^1(G,\mu_{p^\infty}) = k^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p \quad \text{and} \quad H^1(G,\mathbb{Z}_p(1)) = k^\times \mathbin{\widehat{\otimes}} \mathbb{Z}_p, \tag{4.21}$$

where $\mu_{p^\infty} \coloneqq \varinjlim_n \mu_{p^n}$ and $\mathbb{Z}_p(1) \coloneqq \varprojlim_n \mu_{p^n}$.

Example 4.16. Similarly, if E is an elliptic curve over k, then multiplication by p gives an exact sequence

$$0 \to E[p](\bar{k}) \to E(\bar{k}) \to E(\bar{k}) \to 0, \tag{4.22}$$

and the corresponding long exact sequence gives rise to

$$0 \to E(k)/pE(k) \to H^1(G, E[p]) \to H^1(G, E(\bar{k}))[p] \to 0, \tag{4.23}$$

which is the starting point for the definition of Selmer groups, the Tate-Shafarevich group and so on.

Cohomological dimension

Definition 4.17. Let G be a profinite group. We say G has (p-)cohomological dimension at most n if for any (p-)torsion G-module M, we have $H^i(G, M) = 0$ for $i > n$. The (p-)cohomological dimension of G, $\mathrm{cd}(G)$ (resp. $\mathrm{cd}_p(G)$) is the minimal such n.

Given a field k, we tend to write $\mathrm{cd}(k) := \mathrm{cd}(G)$ where $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$.

The following facts are useful in computation. For proofs, see [Ser02, §I.3].

Theorem 4.18. *Let G be any profinite group.*

(i) $\mathrm{cd}(G) = \sup_p cd_p(G)$, *where p runs over the prime numbers.*
(ii) *For any p-Sylow subgroup G_p of G, we have* $\mathrm{cd}_p G = \mathrm{cd}_p G_p$.[15]
(iii) *Let H be a pro-p group.*[16] *Then* $\mathrm{cd}(H) \leq n$ *if and only if* $H^n(H, \mathbb{Z}/p\mathbb{Z}) = 0$.

Corollary 4.19. *For any finite field $\mathbb{F}_q$, we have* $\mathrm{cd}(\mathbb{F}_q) = 1$.

Proof. Note that $G = \mathrm{Gal}(\mathbb{F}_q^{\mathrm{sep}}/\mathbb{F}_q) \cong \widehat{\mathbb{Z}}$. The unique p-Sylow subgroup of $\widehat{\mathbb{Z}}$ is $\mathbb{Z}_p$ (since $\widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$) so by (i) and (ii) we get that

$$\mathrm{cd}(\mathbb{F}_q) = \mathrm{cd}(\widehat{\mathbb{Z}}) = \sup_p \mathrm{cd}_p(\mathbb{Z}_p). \tag{4.24}$$

A standard computation shows that $H^0(\mathbb{Z}_p, \mathbb{Z}/p\mathbb{Z}) = H^1(\mathbb{Z}_p, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. On the other hand, $H^2(\mathbb{Z}_p, \mathbb{Z}/p\mathbb{Z})$ classifies isomorphism classes of extensions

$$0 \to \mathbb{Z}/p\mathbb{Z} \to E \to \mathbb{Z}_p \to 0, \tag{4.25}$$

and by looking at the preimage of the toplogical generator of $\mathbb{Z}_p$, it is not too hard to see that E necessarily splits, that is, $H^2(\mathbb{Z}_p, \mathbb{Z}/p\mathbb{Z}) = 0$, whence our result follows. ☐

[15] For G profinite, a p-Sylow subgroup is a subgroup of maximal index not divisible by p (for more detail, see [Ser02, §I.1.4])

[16] A projective limit of p-groups, groups of order a power of p.

A more difficult result is the following:

Theorem 4.20. *If k is a number field and p prime, then*

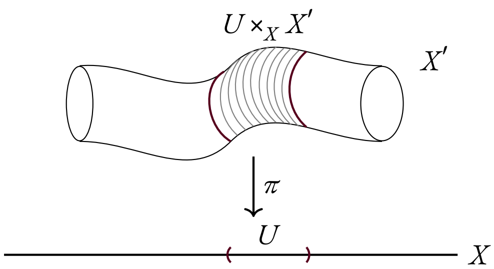
$$\operatorname{cd}_p(k) = \begin{cases} 2 & \text{if} \quad p > 2 \text{ or } k \text{ is totally imaginary,} \\ \infty & \text{if} \quad p = 2 \text{ and } k \text{ is totally real.} \end{cases} \tag{4.26}$$

The issue here is that a real embedding gives rise to an element of order 2 in the Galois group, and by the Hochschild-Serre spectral sequence we can produce a nontrivial element of $H^i(G, \mathbb{Z}/2\mathbb{Z})$ via $H^i(C_2, \mathbb{Z}/2\mathbb{Z})$, where C_2 is the subgroup generated by complex conjugation. See [Ser02, §II.4.4] for a proof.

Higher direct images

We now return to cohomology on sites in general. Recall from section 4.1 that given a continuous map of sites $\pi\colon X'_{E'} \to X_E$, the *higher direct images* are $R^n\pi_*(-)$, the right derived functors of the pushforward $\pi_*\colon \mathrm{Sh}(X'_{E'}) \to \mathrm{Sh}(X_E)$.

Intuitively, higher direct images aim to describe the fibres of the top space X' in terms of the cohomology of the base space. More precisely, if X is a single point, then $R^n\pi_*(-)$ is simply cohomology of the global sections functor. For general X, the idea is that we patch together the cohomology of the fibres X'_x as x varies in X. For example, a theorem of Grothendieck states that under suitably nice conditions, $R^n\pi_*(\mathscr{F})$ vanishes for n greater than the maximal dimension of the fibres.[17]



Proposition 4.21 ([Mil80, III.1.13])**.** *With π as above, $\mathscr{F} \in \mathrm{Sh}(X'_{E'})$, $R^n\pi_*\mathscr{F}$ is isomorphic to the sheafification of the presheaf*

$$U \mapsto H^n(U \times_X X', \mathscr{F}|_{U \times_X X'}). \tag{4.27}$$

Proof. Recall that $\pi_* = a\pi_p i$, where a is the sheafification functor and $i\colon \mathrm{Sh}(X'_{E'}) \hookrightarrow \mathrm{pSh}(X'_{E'})$ is the inclusion functor. From proposition 3.35, π_p is exact, and so is a by proposition 3.40. The main issue is the failure of exactness of i. Now fix

[17] See this M.SE post for a precise statement.

an injective resolution $\mathscr{F} \to \mathscr{I}^\bullet$ – we can always do so since $\mathrm{Sh}(X'_{E'})$ has enough injectives – and note that by exactness,

$$R^n\pi_*\mathscr{F} = H^n(a\pi i I^\bullet) = a\pi_p H^n(iI^\bullet). \tag{4.28}$$

But the presheaf in eq. (4.27) is precisely the one we are applying a to, and this proves our claim. □

With some additional work, one can prove the following stronger result on the small étale site, which states that the cohomology of the fibres is isomorphic to the stalks of the higher direct image functors.

Theorem 4.22 ([Mil80, Thm. III.1.15]). *Let $\pi\colon Y \to X$ be a quasi-compact morphisms of schemes, and $\mathscr{F} \in \mathrm{Sh}(Y_{\text{ét}})$. Let $\bar{x}$ be a geometric point of X, set $i\colon \widetilde{X} := \operatorname{Spec}\mathcal{O}_{X,\bar{x}} \to X$ and $\widetilde{\mathscr{F}} := i^*\mathscr{F}$. Then*

$$(R^n\pi_*\mathscr{F})_{\bar{x}} \cong H^n(Y \times_X \widetilde{X}, \widetilde{\mathscr{F}}). \tag{4.29}$$

Proof sketch. The idea of the proof is to reduce to the case of U affine, then use what Milne calls a "highly technical result" from EGA which allows us to pass the limit inside the cohomology groups. Then

$$\begin{aligned}(R^n\pi_*\mathscr{F})_{\bar{x}} &\cong \varinjlim_U H^n(U\times_X Y, \mathscr{F}|_{U\times_X Y}) \\ &\cong H^n(\varinjlim_U U\times_X Y, \varinjlim_U \mathscr{F}|_{U\times_X Y}) \\ &= H^n(Y\times_X \widetilde{X}, \widetilde{\mathscr{F}}),\end{aligned} \tag{4.30}$$

where the first isomorphism follows from the preceding proposition, and the second is the highly technical result. □

The following is one of the most famous applications of the Grothendieck spectral sequence:

Theorem 4.23 (Leray spectral sequence). *Let $\pi\colon X'_{E'} \to X_E$ be a continuous map of sites, and $\mathscr{F} \in \mathrm{Sh}(X'_{E'})$. Then we have a spectral sequence beginning on the second page,*

$$H^p(X_E, R^q\pi_*\mathscr{F}) \Rightarrow H^{p+q}(X'_{E'}, \mathscr{F}) \tag{4.31}$$

Proof. We give a quick proof in the case where $X_E = X_{\text{ét}}$, and refer to [Mil80, Thm. III.1.18] for (not much) more detail. For the étale site, π_* has an exact left adjoint so preserves injectives. Thus it satisfies the conditions for the Grothendieck spectral sequence (theorem 4.12), which immediately gives the result. □

4.4 Cohomology with supports

Speakers: George Robinson and George Cooper

One important application of the six functor setup of section 3.5 is to define an algebro-geometric analogue of the theory of cohomology with compact support for manifolds. For the remainder of the section, let $i\colon Z \to X$ be a closed immersion, and $j\colon U = X \setminus Z \to X$ be the corresponding open immersion.

Definition 4.24. The functor

$$\Gamma(Z, i^!(-))\colon \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Ab}, \qquad \mathscr{F} \mapsto \ker\big(\mathscr{F}(X) \to \mathscr{F}(U)\big) \tag{4.32}$$

is left exact, and $H^n_Z(-) := R^n\Gamma(Z, i^!(-))$ is called **the n-th cohomology with support in Z.**

As the name suggests, $H^n_Z(-)$ is a cohomological delta-functor. Cohomology with support in Z relates to usual sheaf cohomology as follows:

Lemma 4.25 ([Mil80, Prop. III.1.25])**.** *With notation as above, there exists a long exact sequence*

$$\ldots \to H^r_Z(X_{\text{ét}}, \mathscr{F}) \to H^r(X_{\text{ét}}, \mathscr{F}) \to H^r(U_{\text{ét}}, \mathscr{F}) \to H^{r+1}_Z(X_{\text{ét}}, \mathscr{F}) \to \ldots, \tag{4.33}$$

which is natural in X, Z and $\mathscr{F}$.

Proof. There is a natural isomorphism $\mathrm{Hom}_X(\underline{\mathbb{Z}}_X \mathscr{F}) \xrightarrow{\sim} \Gamma(X_{\text{ét}}, i^!\mathscr{F})$. In particular, $H^p_Z(X_{\text{ét}}, -) \cong \mathrm{Ext}^p_X(\underline{\mathbb{Z}}_X, -)$.

Now, recall that we have an adjunction between j^* and $j_!$, $\mathrm{Hom}_X(j_!j^*\underline{\mathbb{Z}}_X, \mathscr{F}) \cong \mathrm{Hom}_X(j^*\underline{\mathbb{Z}}_X, j^*\mathscr{F})$, so $\mathrm{Ext}_X(j_!h^*\underline{\mathbb{Z}}_X, \mathscr{F}) \cong H^r(U_{\text{ét}}, \mathscr{F})$. We have a short exact sequence (see [Mil80, Rmk. II.3.13]),

$$0 \to j_!j^*\underline{\mathbb{Z}}_X \to \underline{\mathbb{Z}}_X \to i_*i^*\underline{\mathbb{Z}}_X \to 0, \tag{4.34}$$

and since Hom_X is left exact, the sequence

$$0 \to \mathrm{Hom}_X(i_*i^*\underline{\mathbb{Z}}_X, \mathscr{F}) \to \mathrm{Hom}_X(\underline{\mathbb{Z}}_X, \mathscr{F}) \to \mathrm{Hom}_X(j_!j^*\underline{\mathbb{Z}}_X, \mathscr{F}) \tag{4.35}$$

is exact. Therefore we have isomorphisms $\mathrm{Hom}_X(i_*i^*\underline{\mathbb{Z}}_X, \mathscr{F}) \cong \Gamma(X_{\text{ét}}, i^!\mathscr{F})$, so the long exact sequence in $\mathrm{Ext}^\bullet$ gives the corresponding for $H^\bullet_Z$, proving our claim. □

Theorem 4.26 (The Excision Theorem)**.** *Let $\pi\colon X' \to X$ be étale, $Z' \subset X'$ be closed and assume*

(i) *$Z := \pi(Z')$ is closed, and the restriction $\pi|_{Z'} : Z' \to Z$ is an isomorphism;*

(ii) $\pi(X' \setminus Z') \subset X \setminus Z$.

Then for any $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$, we have $H^r_Z(X_{\text{ét}}, \mathcal{F}) \cong H^r_{Z'}(X'_{\text{ét}}\mathcal{F}|_{X'})$.

Informally, this says that if we modify X away from Z, the cohomology with support in Z, H^r_Z, is unchanged. This is an algebro-geometric version of a similar statement from algebraic topology.

Proof. By assumption (i), we have a commutative diagram

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d] & X' \arrow[d, "\pi"] & Z' \arrow[l, "i'"'] \arrow[d, "\cong"] \\ U \arrow[r, "j"] & X & Z \arrow[l, "i"'] \end{tikzcd} \tag{4.36}$$

where the undefined objects and maps are the natural ones. From this and the exact sequence in eq. (4.34) (with $\mathcal{F}$ in place of $\underline{\mathbb{Z}}_X$) the diagram

$$\begin{tikzcd} 0 \arrow[r] & \Gamma_{Z'}(X', \pi^*\mathcal{F}) \arrow[r] & \Gamma(X', \pi^*\mathcal{F}) \arrow[r] & \Gamma(U', \pi^*\mathcal{F}) \\ 0 \arrow[r] & \Gamma_Z(X, \mathcal{F}) \arrow[r] \arrow[u, "\phi"] & \Gamma(X, \mathcal{F}) \arrow[r] \arrow[u] & \Gamma(U, \mathcal{F}) \arrow[u] \end{tikzcd} \tag{4.37}$$

commutes. Since π^* is exact and preserves injectives, it suffices to prove the statement for $r = 0$, which amounts to showing that ϕ is an isomorphism.

Let's first prove that ϕ is injective: If $s \in \Gamma_Z(X, \mathcal{F})$ maps to 0, then s restricts to 0 in $\Gamma_Z(X', \mathcal{F})$ and also in $\Gamma(U, \mathcal{F})$, since s is supported on Z. Since $\{X' \to X, U \to X\}$ is an étale cover, $s = 0$ by the sheaf condition.

Next we show surjectivity: if $s' \in \Gamma_{Z'}(X', \pi^*\mathcal{F})$, then the idea is to glue the image of s' in $\Gamma(X', \pi^*\mathcal{F})$ and $0 \in \Gamma(U, \mathcal{F})$ to obtain an element of $\Gamma(X, \mathcal{F})$ which vanishes outside Z, hence pulls back to $\Gamma_Z(X, \mathcal{F})$. But the two agree on $X' \times_X U \subset U'$, and so indeed glue to a global section on X. □

Corollary 4.27. *If $x \in X$ is a closed point and $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$, then we have isomorphisms $H^r_{\{x\}}(X, \mathcal{F}) \cong H^r(\operatorname{Spec} \mathcal{O}^{\text{sh}}_{X,x}, \mathcal{F})$.*

Proof. Apply the theorem to étale neighbourhoods of x, and take the limit using [Mil80, Lemma III.1.16]. □

Bibliography

[AM94] M.F. Atiyah and I.G. MacDonald. *Introduction to Commutative Algebra*. Addison-Wesley Series in Mathematics. Avalon Publishing, 1994. 9, 10, 12, 13

[Cho06] Timothy Y. Chow. You could have invented spectral sequences. *Notices of the American Mathematical Society*, 53(1):15–19, 2006. 5

[Har77] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1977. 5, 7, 11, 12, 16, 17, 35, 52

[KS13] Masaki Kashiwara and Pierre Schapira. *Sheaves on Manifolds: With a Short History. «Les Débuts de La Théorie Des Faisceaux». By Christian Houzel*. Springer Science & Business Media, March 2013.

[LE06] Q. Liu and R. Erne. *Algebraic Geometry and Arithmetic Curves*. Oxford Graduate Texts in Mathematics (0-19-961947-6). Oxford University Press, 2006. 15

[Mil80] James S. Milne. *Etale Cohomology (PMS-33)*. Princeton University Press, 1980. 8, 9, 10, 11, 14, 16, 19, 24, 31, 32, 33, 34, 35, 43, 44, 45, 50, 51, 55, 56, 57, 59

[MR89] H. Matsumura and M. Reid. *Commutative Ring Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1989. 20

[Mum67] David Mumford. *Introduction to Algebraic Geometry*. Department of Mathematics, Harvard University, 1967. 11, 12, 16, 20

[NS13] J. Neukirch and N. Schappacher. *Algebraic Number Theory*. Grundlehren Der Mathematischen Wissenschaften. Springer Berlin Heidelberg, 2013. 17

[Ser73] Jean-Pierre Serre. *A Course in Arithmetic*. Graduate Texts in Mathematics. Springer-Verlag, New York, 1973. 54

[Ser95] Jean-Pierre Serre. *Local Fields*. Graduate Texts in Mathematics. Springer New York, 1995. 54

[Ser02] Jean-Pierre Serre. *Galois Cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2002. 42, 43, 54

[Sha13] Igor R. Shafarevich. *Basic Algebraic Geometry 2: Schemes and Complex Manifolds*. Springer-Verlag, Berlin Heidelberg, third edition, 2013. 56

[Sta21] The Stacks project authors. The stacks project. 2021. 19, 23, 27, 30, 32, 54, 55, 59

[Wei94] Charles A Weibel. *An Introduction to Homological Algebra*. Number 38. Cambridge university press, 1994. 40