Proper base change II

Étale cohomology study group

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Review

Main goal:

Theorem

Base change works for any proper morphism $f : X \to Y$*, i.e. for any torsion étale sheaf* $\mathcal{F}$ *on* X *and morphism* $g : Y' \to Y$*, we have* $g^*(R^n f_* \mathcal{F}) \xrightarrow{\cong} R^n f'_*(g'^* \mathcal{F})$.

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Key case:

Proposition

Let A be a strictly henselian local ring, $X = \mathbb{P}^1_A$, X_0 the special fiber. Then $H^n_{ét}(X, \mathcal{F}) \cong H^n_{ét}(X_0, \mathcal{F}|_{X_0})$.

Review 2: Injective Boogaloo

Key reduction (follows from the $n = 0$ case):

Lemma

Base change holds for $f : X \to Y$ iff for all injective $\mathbb{Z}/\ell$-sheaves $\mathcal{I}$ on X_{et} and $g : Y' \to Y$, $g'^{-1}\mathcal{I}$ is f'_*-acyclic: $R^n f'_*(g'^{-1}\mathcal{I}) = 0$ for all $n > 0$.

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Therefore it is enough to show:

Proposition

Let A be a henselian local ring, $X = \mathbb{P}^1_A, X_0$ the special fiber. For any injective étale $\mathbb{Z}/\ell$-module $\mathcal{I}$ on X, $H^n_{\acute{e}t}(X_0, \mathcal{I}|_{X_0}) = 0$ for $n > 0$.

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In general ζ need not lift, but we will modify $\mathcal{F}$ to achieve that.

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By Leray spectral sequence + vanishing of $R^n f_*$ (since f is finite), we have

$$H^1_{ét}(X, f_*\underline{M}) \cong H^1_{ét}(Y, \underline{M}),$$
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Hence classes from $H^1_{ét}(X_0, f_*\underline{M}|_{X_0})$ lift to $H^1_{ét}(X, f_*\underline{M})$ and we can proceed as before.

Covers by affines and vanishing of cohomology

We want to show that given an injective sheaf on $X = \mathbb{P}^1_A$, we have $H^n_{ét}(X_0, \mathcal{I}|_{X_0}) = 0$ for injective $\mathbb{Z}/\ell$-module $\mathcal{I}$ for $n > 1$.

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Theorem

Let X be a separated scheme covered by $k+1$ affine opens, $Z \subseteq X$ closed subscheme, $\mathcal{I}$ an injective étale $\mathbb{Z}/\ell$-module. Then $H^n_{ét}(Z, \mathcal{I}|_Z) = 0$ for $n > k$.

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The hard part is $k = 0$, the rest follows by induction using Mayer-Vietoris exact sequence: if $X = U \cup V$, U affine, V union of k affines, then we have $U \cap V$ union of k affines and

$$0 = H^{n-1}_{ét}(U \cap V \cap Z, \mathcal{I}|_Z) \to H^n_{ét}(Z, \mathcal{I}|_Z) \to H^n_{ét}(U \cap Z, \mathcal{I}|_Z) \oplus H^n_{ét}(V \cap Z, \mathcal{I}|_Z) = 0$$

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Let (A, I) be a henselian pair, $X = \operatorname{Spec} A$, $Z = \operatorname{Spec} A/I$. For any torsion étale sheaf $\mathcal{F}$ on X we have $H^n_{ét}(X, \mathcal{F}) \cong H^n_{ét}(Z, \mathcal{F}|_Z)$.

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$$\begin{array}{ccccccc} H^{n-1}_{ét}(X, \mathcal{F}') & \longrightarrow & H^{n-1}_{ét}(X, \mathcal{F}'') & \longrightarrow & H^n_{ét}(X, \mathcal{F}) & \longrightarrow & H^n_{ét}(X, \mathcal{F}') \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{n-1}_{ét}(Z, \mathcal{F}'|_Z) & \to & H^{n-1}_{ét}(Z, \mathcal{F}''|_Z) & \to & H^n_{ét}(Z, \mathcal{F}|_Z) & \to & H^n_{ét}(Z, \mathcal{F}'|_Z). \end{array}$$

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Let $X = \operatorname{Spec} A$ any affine scheme, $Z = \operatorname{Spec} A/I$ a closed subscheme, $\mathcal{I}$ an injective torsion étale sheaf. We want $H^n_{\acute{e}t}(Z, \mathcal{I}|_Z) = 0$ for $n > 0$.

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Let $f : X \to Y$ be proper. If all (geometric) fibers of f have dimension $\leq d$, then for all torsion étale sheaves $\mathcal{F}$ we have $R^n f_* \mathcal{F} = 0$ for $n > 2d$.

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Proof.

By proper base change, for all geometric points $\overline{y}$ we have

$$(R^n f_* \mathcal{F})_{\overline{y}} = H^n(X_{\overline{y}}, \mathcal{F}_{\overline{y}}).$$

We are then done by results on cohomological dimension. □

Corollary: invariance under base field extension

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Let $f : X \to \operatorname{Spec} k$ be a proper variety over separably closed k, and K separably closed extension of k. For any torsion étale sheaf $\mathcal{F}$ on X we have an isomorphism

$$H^n_{ét}(X_K, \mathcal{F}_K) \cong H^n_{ét}(X, \mathcal{F}),$$

where X_K is the base change of X and $\mathcal{F}_K$ the corresponding pullback.

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Proof.

By proper base change, these coincide with the stalks of $R^n f_* \mathcal{F}$ at geometric points corresponding to k, K. Since k is separably closed, both are just global sections. □

Corollary: cohomology with compact support

Recall: for an open immersion $j : U \hookrightarrow X$, let $j_!$ be the extension by zero functor $\mathrm{Sh}(U_{\acute{e}t}) \to \mathrm{Sh}(X_{\acute{e}t})$.

Definition

For an étale sheaf $\mathcal{F}$ on U, we define *cohomology with compact support* as $H^n_{\acute{e}t,c}(U, \mathcal{F}) = H^n_{\acute{e}t,c}(X, j_!\mathcal{F})$ for any inclusion $j : U \hookrightarrow X$ into a proper scheme.

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More generally:

Definition

Let $\pi : U \to S$ be compactifiable, meaning there exists an open immersion $j : U \to X$ into a proper $\overline{\pi} : X \to S$. Define *higher direct image with compact support* as $R^n_c\pi_*\mathcal{F} = R^n\overline{\pi}_*j_!\mathcal{F}$.

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The cohomology sheaves $R^n_c\pi_*\mathcal{F}$ are independent of the compactification $j : U \hookrightarrow X$.

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Let $\overline{\pi} : X \to S, \overline{\pi}' : X \to S$ be proper and $j : U \to X, j' : U \to X'$ open immersions.

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$$(R^p \overline{\pi}_*)(R^q g_*)(j'_! \mathcal{F}) \Rightarrow (R^{p+q} \overline{\pi}'_*)(j'_! \mathcal{F}).$$

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Then, for the constant sheaf $\mathcal{F} = \mathbb{Z}_X$, we have

- $R^1 f_* \mathbb{Z}_X = 0$,
- $H^1(X_0, \mathbb{Z}_{X_0}) \neq 0$.

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On the other hand, $H^1(X_0, \mathbb{Z}_{X_0}) = \mathbb{Z}$: for the double point Q, the fiber of $i_*\mathbb{Z}_{x_0}$ at Q is $\mathbb{Z}^2$, so we have $0 \to \mathbb{Z}_{X_0} \to i_*\mathbb{Z}_{x_0} \to \mathbb{Z}_Q \to 0$. We then have

$$0 \to H^0(X_0, \mathbb{Z}_Q) \to H^1(X_0, \mathbb{Z}_{X_0}) \to 0$$

and so $H^0(X_0, \mathbb{Z}_Q) = \mathbb{Z} \neq 0$.