Proper base change I

Étale cohomology study group

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Review: edge maps in spectral sequences

Theorem (Grothendieck spectral sequence)

Suppose $A \xrightarrow{G} B \xrightarrow{F} C$ is a chain of functors between categories. Under appropriate conditions, for any $a \in A$ we have a spectral sequence

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In our case of interest these are the maps

$$R^n f_* \circ g'_* \to R^n(f_* g'_*) = R^n(g_* f'_*) \to g_* \circ R^n f'_*.$$

Construction of the base change map

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Like before, we have $\mathcal{F} \to g'_* g'^* \mathcal{F}$ and hence $R^n f_* \mathcal{F} \to R^n f_*(g'_* g'^* \mathcal{F})$. We now use the following map $R^n f_* \circ g'_* \to g_* \circ R^n f'_*$ constructed before: $R^n f_*(g'_* g'^* \mathcal{F}) \to g_* R^n f'_*(g'^* \mathcal{F})$.

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Suppose $f : X \to Y$ is a proper morphism and $\mathcal{F}$ is a torsion étale sheaf. Then the base change morphism is an isomorphism.

There are other conditions which imply this isomorphism, e.g. g being smooth (assuming extra technical conditions.)

Illustration: topological base change

Suppose we have a Cartesian diagram of topological spaces:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

We have to check the isomorphism on stalks: for all $y \in Y'$ we want an isomorphism

$$(g^*(R^n f_* \mathcal{F}))_y = (R^n f_* \mathcal{F})_{g(y)} \cong (R^n f'_*(g'^* \mathcal{F}))_y.$$

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Observe that the *fibers* $X_{g(y)}$ and X'_y are isomorphic, and the isomorphism identifies $\mathcal{F}|_{X_{g(y)}}$ with $(g'^*\mathcal{F})|_{X'_y}$. It would be enough to have an isomorphism $(R^n f_* \mathcal{F})_{g(y)} \cong H^n(X_{g(y)}, \mathcal{F}|_{X_{g(y)}})$—base change for inclusion of a point.

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You can prove this for higher cohomology too. See Milne's *Lectures on Étale Cohomology*, Section 17.

Sections on fibers

Lemma

Let $(A, \mathfrak{m})$ *be a henselian local ring,* $X \to \mathrm{Spec}(A)$ *proper,* Z *the special fiber. For any (Zariski) sheaf* $\mathcal{F}$ *on* X *we have* $\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z)$.

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Sections on geometric fibers

Proposition

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$f : X \to Y$ *proper,* $\overline{y}$ *a geometric point of* Y*. For any étale sheaf* $\mathcal{F}$ *we have an isomorphism* $(f_*\mathcal{F})_{\overline{y}} \to \Gamma(X_{\overline{y}}, \mathcal{F}_{\overline{y}})$.

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Corollary (Base change in degree 0)

Base change map $g^{-1}f_*\mathcal{F} \to f'_*g'^*\mathcal{F}$ *is an isomorphism.*

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We say *base change holds* for proper $f : X \to Y$ if for any $g : Y' \to Y$ and torsion étale sheaf $\mathcal{F}$ on X, base change map $g^{-1}R^n f_* \mathcal{F} \to R^n f'_* g'^* \mathcal{F}$ is an isomorphism for all n.

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Base change holds for f iff for all injective $\mathbb{Z}/\ell$-sheaves $\mathcal{I}$ on X_{et} and $g : Y' \to Y$, $g'^{-1}\mathcal{I}$ is f'_*-acyclic: $R^n f'_*(g'^{-1}\mathcal{I}) = 0$.

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Proof.

One direction clear. For the converse, use colimits and induction to go from $\mathbb{Z}/\ell$ to arbitrary torsion. Then use acyclic resolutions $g'^{-1}\mathcal{I}^\bullet$ to compute cohomology and use $n = 0$ case. □

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Proposition

If f_1 is surjective and base change works for $f_1, f_2 \circ f_1$, then it works for f_2.

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On $S = \operatorname{Spec} A_i$, we have $\Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{s}}$, which proves $\Gamma(S, -)$ is exact, so higher cohomology vanishes. □

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Let $f : X \to Y$ be finite. Then $R^n\mathcal{F} = 0$ for all $n > 0$ and all étale sheaves $\mathcal{F}$ on X.

Proof.

Let $X^{sh} = X \times_Y \mathcal{O}^{sh}_{Y,\overline{y}}$. Then $(R^n\mathcal{F})_{\overline{y}} = H^n(X^{sh}, \mathcal{F})$. We have $X^{sh} = \operatorname{Spec} A$, where A is a finite $\mathcal{O}^{sh}_{Y,\overline{y}}$-algebra. Hence $A = \prod A_i$ with A_i strictly henselian.

On $S = \operatorname{Spec} A_i$, we have $\Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{s}}$, which proves $\Gamma(S, -)$ is exact, so higher cohomology vanishes. □

Corollary

Base change works for finite morphisms.

Reduction to $\mathbb{P}^1$

Proposition

To prove base change for all proper $f : X \to Y$, it is enough to show it works for $\mathbb{P}^1_Y \to Y$.

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Can assume Y affine. Chow's lemma:

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Hence enough to show for $\mathbb{P}^n_Y \to Y$. We have a finite surjective map $(\mathbb{P}^1_Y)^n \to \mathbb{P}^n_Y$. Finally we can reduce to $\mathbb{P}^1$ by induction. □