The Weil conjectures

Étale cohomology reading seminar

30/07/21

Outline

Statement of Weil conjectures

W1 and W2, “rationality” and “integrality”

W5: “Functoriality

W3: “Functional equation”

Summary of the étale cohomology seminar

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- eg. for $X = \mathbb{P}^n_{\mathbb{F}_q}$,

$$Z(X,t) = \frac{1}{(1-t)(1-qt)\dots(1-q^n t)}$$

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(W5) “Functoriality”: if $X = X_0 \times \operatorname{Spec} \mathbb{F}_q$ for some X_0/k a nr. field, then

$$\deg P_i = \beta_i(X_0) \coloneqq \dim H^i(X_0(\mathbb{C}), \mathbb{C})$$

W1 and W2, “rationality” and “integrality”

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- We have proved (W1)! Link.

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- **Cor.** $a_{i,r} \in \overline{\mathbb{Q}}$, and we have proved (W2):

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- **Claim:** $\dim H^r_{\text{ét}}(\overline{X}; \mathbb{Q}_\ell) \cong \dim H^r(X_0(\mathbb{C}); \mathbb{C})$. **Pf:**
  - $\mathbb{F}_q \cong \mathscr{O}_k/\mathfrak{p}$ for some $\mathfrak{p}$ over p, and $\widehat{\mathscr{O}_{k,\mathfrak{p}}}$ has generic fibre $X_\eta := X_0 \times \operatorname{Spec}(\mathbb{Q}_p \otimes k)$ and special fibre X_0.

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  - By cor. of smooth base change, ($\overline{\eta}$ geom. pt. over η)

$$H^\bullet_{\text{ét}}(\overline{X}; \mathbb{Q}_\ell) \cong H^\bullet_{\text{ét}}(X_{\overline{\eta}}; \mathbb{Q}_\ell).$$

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$$Z(X, 1/(q^d t)) = \pm q^{d\chi/2} t^{\chi} Z(X, t).$$

[1][Mil80] p.289

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$= q^d \eta(x \smile y)$, so $\lambda_{i,2d-r} = q^d / \lambda_{i,r}$ (up to conjugation, since η is defined over $\mathbb{Q}_\ell$).

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$$\begin{aligned} Z(t) = \prod_{r,i}(1 - ta_{i,r})^{(-1)^{i+1}} &= \prod_{r,i}\left(-\frac{tq^d}{a_{i,2d-r}}\left(1 - \frac{a_{i,2d-r}}{tq^d}\right)\right)^{(-1)^{i+1}} \\ &= \prod_{r,i}\left(-\frac{tq^d}{a_{i,2d-r}}\right)^{(-1)^{i+1}} \prod_{r,i}\left(1 - \frac{a_{i,2d-r}}{tq^d}\right)^{(-1)^{i+1}} \\ &= \prod_{r,i}\left(-\frac{tq^d}{a_{i,2d-r}}\right)^{(-1)^{i+1}} \cdot Z(1/(q^d t)). \end{aligned}$$

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$$\pm 1 = \begin{cases} 1 & \text{if } d \text{ is even,} \\ (-1)^N & \text{otherwise.} \end{cases}$$

Weil conjectures

(W1) "Rationality":

$$Z(X,t) = \frac{P_1(t)\dots P_{2d-1}(t)}{P_0(t)\dots P_{2d}(t)}.$$

(W2) "Integrality":

$$P_0(t) = 1 - t, \quad P_{2d}(t) = 1 - q^d t, \quad P_r(t) = \prod_i (1 - a_{i,r}t), \qquad a_{i,r} \in \overline{\mathbb{Q}}.$$

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(W4) "Riemann hypothesis": The numbers $a_{i,r}$ are Weil numbers, i.e. all their conjugates have real absolute value $q^{r/2}$.

(W5) "Functoriality": if $X = X_0 \times \operatorname{Spec} \mathbb{F}_q$ for some X_0/k, then

$$\deg P_i = \beta_i(X_0) := \dim H^i(X_0(\mathbb{C}), \mathbb{C})$$

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Trace lemma

back See [Mil00, Lemma 27.5] for a somewhat dubious proof; otherwise, see [Del74, (1.5.3)] for a very sleek but not very informative proof.