ÉTALE COHOMOLOGY SEMINAR: PROGRAM

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Except where indicated we aim for each topic to be covered over the course of one week – which translates to two talks of one hour each. These may be given by one and the same person or by different persons. Our main reference is [Mil80]. An excellent guide through that book is also [Mil13].

1. Introduction

Motivation for studying étale cohomology and overview of topics to be covered (1 talk only)

2. Étale morphisms

[Mil80, I.1–3]

- state our running hypothesis throughout: all rings are noetherian, all schemes are locally noetherian
- recall finite and quasi-finite morphisms, give (non-)examples illustrating the difference, state Zariski's main theorem (without proof) and give [Mil80, I.1.10] as corollary
- define flat morphisms of rings and give (non-)examples; you should cover at least [Mil80, I.2.9, 2.12, 2.16]
- main definition is that of étale morphism: you should spend some time exploring consequences, (non-)examples; discuss [Mil80, I.3.4] at length, in particular explain why between non-singular complex algebraic varieties, a morphism is étale if and only if the associated holonomic map between complex manifolds is a local isomorphism; let d be a square-free integer and discuss at which points the induced morphism $\mathrm{Spec}(\mathbb{Z}[\sqrt{d}]) \to \mathrm{Spec}(\mathbb{Z})$ is étale (draw pictures!); you should cover at least [Mil80, I.3.6, 3.8, 3.14, 3.16]

3. Étale Sheaves

[Mil80, II.1, I.4]

*Last update: July 25, 2021

- define small étale site (compare it with small Zariski site);[1] recall/define presheaves and sheaves with values in sets and abelian groups; give lots of examples, including constant and locally constant sheaves, representable sheaves, the structure sheaf $\mathcal{O}$; explain the equivalence of [Mil80, II.1.9] in detail
- describe stalks in terms of étale neighborhoods and the action of the absolute Galois group; compute the stalks $\mathcal{O}_{X,\bar{x}}$ of the structure sheaf: these are 'local rings for the étale topology', state their main properties (strictly Henselian rings) from [Mil80, I.4], including [Mil80, I.4.2abc, 4.4, 4.5], give examples
- show that a sequence of étale sheaves is exact if and only if it so on stalks, cf. [Mil80, II.2.17]

4. Operations on sheaves

[Mil80, II.2–3][2]

- define a continuous morphism of sites and the associated direct and inverse images at the presheaf level; give the explicit formula for the inverse image [Mil80, proof of II.2.2], deduce that inverse images are exact; describe sheafification and give [Mil80, II.2.15]; explain Kummer and Artin-Schreier sequences [Mil80, II.2.18]
- define direct and inverse images at the level of étale sheaves, note again that inverse image is exact; you should cover at least [Mil80, II.3.2, 3.5, 3.6]; define extension-by-zero along open immersions (for closed immersions this is just the direct image); talk about situation around [Mil80, II.3.10]; briefly mention construction of tensor product and internal hom

5. Cohomology

[Mil80, III.1]

- most will have encountered derived functors before so you can be quick in the first results of the section; define the étale cohomology groups and higher direct images; discuss [Mil80, III.1.7a], including some sample computations, say with K a finite field or a number field;[3] [Mil80, III.1.13] is useful; deduce [Mil80, III.1.15] and do [Mil13, 12.4, 12.5]
- explain [Mil80, III.1.18] in the case of étale sites (see [Mil80, III.1.20a]), many will not have encountered spectral sequences before so you should give more details [Mil80, Appendix B.1]; apply this to [Mil13, 12.8]
- define cohomology with supports, you should cover at least [Mil80, III.1.25, 1.27, 1.29]

[1] note that what Milne calls Grothendieck topology [Mil80, II.1.1] is usually called a Grothendieck *pre*-topology
[2] whenever it makes sense you should assume that the site in question is the small étale site of a scheme
[3] these are important examples: later on we will see that the cohomological dimension of a finite type scheme over a field is bounded by twice its Krull dimension + the cohomological dimension of the field

6. First computations

[Mil80, III.2–3]

- briefly recall Čech cohomology and give the main comparison results with étale cohomology, particularly [Mil80, III.2.17]; interpret the first Čech cohomology group at least with coefficients in $\mathbb{G}_m$, that is, state Hilbert's Theorem 90 [Mil80, III.4.9] (do not give a proof in this generality; you will give one in a special case below)
- discuss [Mil80, III.2.22] in detail: the Weil-divisor exact sequence and the cohomology of the multiplicative group on some schemes; show $\mathrm{H}^1(X, \mathbb{G}_m) = \mathrm{Pic}(X)$ and discuss at least the cases (a), (b), (d) of *loc. cit.*
- state [Mil80, III.3.12] and explain at least the main ingredients of the proof (Artin neighborhoods, Riemann existence theorem), explain why we have to restrict to finite coefficients (if nobody else has done that in a previous talk)

7. Cohomology of curves

[Mil80, V.1–2]

- recall the notion of locally constant sheaves and note that inverse image functors preserve them, but direct image functors do not; this motivates the more general notion of *constructible sheaves*: fix a noetherian scheme X; an étale sheaf F of abelian groups (resp. $\mathbb{Z}/n$-modules) on X is constructible if there is a finite partition $X = \amalg X_i$ with X_i locally closed subschemes such that $F|_{X_i}$ is finite locally constant (that is, locally constant with finite stalks).[4] This is in fact a Zariski local condition.[5] State (without proof) that the category of constructible sheaves (as a full subcategory of $\mathrm{Sh}(X_{\text{ét}})$ resp. $\mathrm{Sh}(X_{\text{ét}}, \mathbb{Z}/n)$) is very nice, for example it is closed under kernels, cokernels, extensions, tensor products (in particular, the category is abelian). You may find [Sta21, Tag 05BE] helpful in all of this. Finally [Mil80, p. 163], assume n is invertible in every residue field of X and define $F(r)$ for $r \in \mathbb{Z}$, recall that it is noncanonically locally isomorphic to F (exercise 1 from week 3).
- State the main theorem of this section, [Mil80, V.2.1], and explain the statement: cohomology with compact support from [Mil80, III.1.29] and the canonical 'cup product' pairing of Ext groups from [Mil80, V.I] (you can be brief here: this pairing is completely formal). Give an outline of the proof (in particular, explain where the trace map $\mathrm{H}^2_c(U, \mu_n) \cong \mathbb{Z}/n$ comes from and point out how we use the computations from Section 6)

[4] Milne uses a different (but equivalent, see [Mil80, V.1.8]) definition of constructible sheaves – please don't use that one.

[5] And this allows one to extend the definition of constructible sheaves to arbitrary (always: locally Noetherian) schemes in a reasonable way.

- Applications: describe the cohomology ring of a smooth projective curve over an algebraically closed field entirely (see also [Mil80, V.2.4(f)]), compare with situation over non-algebraically closed fields, for example [Mil80, V.2.3], and possibly with non-projective curves, maybe say something about Artin–Verdier duality [Mil80, V.2.4(d)]

8. Cohomological dimension

[Mil80, VI.1] *(1 talk only)*

- in Section 5 we defined the cohomological dimension of fields; generalize this notion to schemes [Mil80, VI.1] and prove [Mil80, VI.1.4]; given the results in Section 5 we have bounds on cohomological dimension of schemes over separably closed fields, finite fields, and number fields, among others. Also make remark [Mil80, VI.1.5(b)].

9. Proper base change

[Sta21, Tag 095S], [Mil80, VI.2]

- Give a sketch of the proof of the proper base change theorem [Mil80, VI.2.3] following [Sta21, Tag 095S]; do not use the language of derived categories though (for the definition of the base change morphism without derived categories see [Mil80, p. 223])
- deduce [Mil80, VI.2.5–6, 3.1]; the last of these justifies our earlier 'definition' of cohohomology with compact support
- if time remains discuss the failure of proper base change for non-torsion coefficients [Mil80, VI.2.4]

10. Smooth base change

[Mil80, VI.4] *(1 talk only)*

- sketch the proof of [Mil80, VI.4.1]
- deduce [Mil80, VI.4.3] and compare with the earlier [Mil80, VI.2.6] (see [Mil80, VI.4.4])

11. Finiteness theorem

Given that I will talk, this is only sketched. (1 talk only)

- state that, under suitable assumptions, higher direct image functors preserve constructible sheaves, and along smooth proper morphisms even locally constant constructible ones (compare with topological analogue); we'll see how much of the proof we will be able to give
- many applications: [Mil80, VI.2.8] (for separably closed or finite fields; note that the statement in the book is false in general), [Mil80, VI.3.2(d)] (and maybe more general remark about 'six functors'), [Mil80, VI.4.2]

12. Cohomological purity, cycle classes

[Mil80, VI.5, 6, (7), 9, 10], [Mil13, § 23]

- State the absolute cohomological purity theorem in the form of [Fuj02, p. 153] (due to Gabber): *Let $i : Z \hookrightarrow X$ be a closed immersion of pure codimension c between regular $\mathbb{Z}[1/n]$-schemes. Then* $\underline{\mathrm{H}}^q_Z(X, \mathbb{Z}/n) := \mathrm{R}^q i^! \mathbb{Z}/n = \begin{cases} 0 : & q \neq 2c \\ \mathbb{Z}/n(-c) : & q = 2c \end{cases}$
  (At the level of derived categories this takes the more transparent form $\mathrm{R}i^!\mathbb{Z}/n(c)[2c] \cong \mathbb{Z}/n$.) However, we will only discuss the simpler version (proved in [Mil80, VI.5.1]) where X and Z are assumed smooth over a separably closed field $S = \mathrm{Spec}(k)$ and i is an S-morphism.

- Before turning to the proof let us discuss complements and applications:

  1. The isomorphism of the theorem $\mathbb{Z}/n \xrightarrow{\sim} \underline{\mathrm{H}}^{2c}_Z(X, \mathbb{Z}/n(c))$ is induced by a map on global sections: $\mathbb{Z}/n \xrightarrow{\sim} \mathrm{H}^{2c}_Z(X, \mathbb{Z}/n(c))$. This map sends 1 to the *fundamental class* $s_{Z/X}$ of Z in X, see [Mil80, § VI.6]. For example, when Z is a point, this is the analogue of a local orientation in differential topology. Describe the map in detail when $c = 1$, and then explain how, in principle, you can "compute" the class for higher codimension. This is [Mil80, VI.6.1], particularly (c) and (d), but don't state the Theorem in full. (Use that locally i factors as a composite of codimension 1 closed immersions, see also below for this.)
  2. One can replace $\mathbb{Z}/n$ in our 'simplified' theorem by any locally constant $\mathbb{Z}/n$-linear sheaf [Mil80, VI.5.4(b)].
  3. Explain the Gysin sequence [Mil80, VI.5.3] and compute the cohomology of projective space [Mil80, VI.5.6]. If time remains, say something about Weak Lefschetz [Mil80, VI.7.1].

- Give the main ingredients of the proof of [Mil80, VI.5.1]: By proving a more general statement (for arbitrary S) one may do induction on c (explain the geometry here) and thereby reduce to $c = 1$. This case is reduced to S as above and the cohomology of the projective line (which we know well). (It is also worth noting that if the base field is algebraically closed, we computed such cohomology groups in the talks on the cohomology of curves.)

- Recall the Chow ring of algebraic cycles, see [Mil80, § VI.9] and [Mil13, § 23]. Also state the basic properties of the cup product on étale cohomology. Give at least the direct construction of the cycle class map and state [Mil80, VI.10.7]. Depending on the time available, explain ingredients of proof: a second description of the cycle class map via Chern classes. (If you get to discuss that, the projective bundle theorem would be good to state.)

13. Poincaré duality

[Mil80, VI.11] *(1 talk only)*

- Recall Poincaré duality in differential topology as motivation and state the analogue [Mil80, VI.11.2]. Note that the pairing was already constructed in the required generality when we proved PD for curves.
- Explain where the trace map comes from and give a sketch of the proof of PD following [Mil80, VI.11.1].
- (It's doubtful you'll have time but if you do:) Discuss [Mil80, VI.11.5/6]. Both of these are satisfyingly analogous to the situation in differential topology.

14. Towards the Weil conjectures

[Mil80, V.1, VI.12] *(1 talk only)*

- As mentioned in the first talk (parts of) the Weil conjectures should follow from a 'good cohomology theory'. One of the requirements of such a theory is that it takes values in a characteristic zero field. On the other hand, we saw that étale cohomology is well-behaved only for torsion coefficients. The solution to this impediment is *ℓ-adic cohomology*. Follow [Mil80, V.1, p. 163ff] in defining (constructible and lisse) ℓ-adic sheaves, give examples (say, $\mathbb{Z}/\ell^n$-sheaves, or $\mathbb{Z}_\ell(r)$). Define $\mathrm{Cons}(X, \mathbb{Q}_\ell)$ [Mil80, bottom of p. 164] and give [Mil80, V.1.11][6] which allows to reduce certain properties for ℓ-adic sheaves to finite coefficients. Finally, define Betti numbers and Euler characteristics.
- State and prove the Lefschetz trace formula [Mil80, VI.12.3]. This is a nice argument and you should be able to give it in full.

15. Weil conjectures: Rationality, Functional equation

[Mil80, VI.12] *(1 talk only)*

- Briefly recall the Weil conjectures [Mil80, p. 286]; they were stated already (possibly in a slightly different form) in the first talk of the seminar. Prove W.5
- Prove [Mil80, VI.12.4 and 12.6] (corresponding to W1 and W3).

References

[Fuj02] Kazuhiro Fujiwara. "A proof of the absolute purity conjecture (after Gabber)". In: *Algebraic geometry 2000, Azumino (Hotaka)*. Vol. 36. Adv. Stud. Pure Math. Math. Soc. Japan, Tokyo, 2002, 153–183. doi: 10.2969/aspm/03610153.

[6] Give a (sketch of) proof only if you have time.

[Mil13] James S. Milne. *Lectures on Etale Cohomology (v2.21)*. Available at www.jmilne.org/math/. 2013.

[Mil80] James S. Milne. *Étale cohomology*. Vol. 33. Princeton Mathematical Series. Princeton University Press, Princeton, N.J., 1980, xiii+323. ISBN: 0-691-08238-3.

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