Étale cohomology reading seminar

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3.5 Operations on sheaves

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In this section we prove that many of the operations on sheaves generalises from the usual setting. First we need to extend our notion of a continuous map:

Definition 3.31. Let $(\mathrm{C}'/X')_{E'}$ and $(\mathrm{C}/X)_E$ be sites of schemes X and X', and $\pi\colon X' \to X$ a morphism of schemes. We say that π is a **continuous map of sites** if the following conditions hold:

(i) If $Y \in \mathrm{C}$, then $Y_{(X')} := Y \times_X X' \in \mathrm{C}'$.
(ii) In the Cartesian diagram

$$\begin{tikzcd} U_{(X')} \arrow[r] \arrow[d, "f'"] & U \arrow[d, "f"] \\ X' \arrow[r, "\pi"] & X \end{tikzcd} \tag{3.21}$$

if $f \in E$, then the base change f' is in E'.

Here (i) is an analogue of the property that preimages of open sets are open, and by abuse of notation we write $\pi^{-1}(Y) := Y_{(X')} = Y \times_X X'$. On the other hand, (ii) ensures that we don't run into trouble when pulling back covers.

Note that since base change preserves surjectivity (see eg. [Sta21, Lemma 01S1]) a continuous map of sites takes coverings to coverings.

Example 3.32. Any morphism $X' \to X$ induces a continuous map of sites $X'_{\text{ét}} \to X_{\text{ét}}$; this is a direct consequence of proposition 2.44.

Definition 3.33. Let $\pi\colon X'_{E'} \to X_E$ be a continuous map of sites, and $\mathcal{F}'$ a presheaf on $X'_{E'}$. The **direct image presheaf** $\pi_p\mathcal{F}'$ is the presheaf on X_E defined by $\pi_p\mathcal{F}'(U) := \mathcal{F}'(U \times_X X')$.

Note that if $\mathcal{F}'$ is a sheaf, then so is $\pi_p\mathcal{F}'$. In fact, the map π_p is a functor $\mathrm{pSh}(X'_{E'}) \to \mathrm{pSh}(X_E)$. While it is not hard to check that π_p preserves exactness, it is not true for the restriction to the full subcategory of sheaves, $\pi_*\colon \mathrm{Sh}(X'_{E'}) \to \mathrm{Sh}(X_E)$.

As in the case of sheaves on a topological space we can also pull a sheaf on X back along $X' \to X$:

Definition 3.34. Let $\pi : X'_{E'} \to X_E$ be a continuous map of sites. The **inverse image functor** is the functor $\pi^p\colon \mathrm{pSh}(X_E) \to \mathrm{pSh}(X'_{E'})$ given by the left adjoint of π_p.

The existence of such a functor follows from a general category-theoretical argument, and more details can be found in [Mil80, II.2.2].

In the étale topology, we can give a more explicit construction of π^p: For $\mathcal{F} \in \mathrm{pSh}(X_{\text{ét}})$, let $\pi^p\mathcal{F}(U') := \varinjlim_{(g,U)} \mathcal{F}(U)$, where the colimit is taken over pairs (g, U) fitting into a commuting diagram

$$\begin{tikzcd} U' \arrow[r, "g"] \arrow[d] & U \arrow[d] \\ X' \arrow[r, "\pi"] & X \end{tikzcd} \tag{3.22}$$

These form a direct system with morphisms $h\colon U_1 \to U_2$ fitting into the commutative diagrams

$$\begin{tikzcd} U' \arrow[rr, "g_2"] \arrow[dr, "g_1"] \arrow[dd] & & U_2 \arrow[dd] \\ & U_1 \arrow[ur, "h"] \arrow[dr] & \\ X' \arrow[rr, "\pi"] & & X \end{tikzcd} \tag{3.23}$$

One can check by hand that this indeed defines a sheaf on $X'_{\text{ét}}$, and that it is left adjoint to π_p. Note that for general sites X_E and $X'_{E'}$ the functor π^p does not preserve the sheaf condition. See [Mil80, §II.2] for further details.

Proposition 3.35. *Let $\pi\colon X'_{E'} \to X_E$ be a continuous map of sites.*

(i) *The functor $\pi_p\colon \mathrm{pSh}(X'_{E'}) \to \mathrm{pSh}(X_E)$ is exact;*
(ii) *the functor $\pi^p\colon \mathrm{pSh}(X_E) \to \mathrm{pSh}(X'_{E'})$ is right exact;*
(iii) *the functor $\pi^p\colon \mathrm{Sh}(X_E) \to \mathrm{pSh}(X'_{E'})$ is also left exact in the étale topology.*

Proof. (i) follows directly from definition; simply check every étale $U \to X$. (ii) follows from adjointness. Finally, (iii) follows from the fact that $\pi^p\mathcal{F}(U')$ is a *cofiltered colimit* in Ab, and these are exact by a general category-theoretic argument. □

Example 3.36 (The Kummer sequence)**.** Recall that the étale sheaf μ_n from example 3.7 is represented by

$$\operatorname{Spec} \frac{\mathbb{Z}[t, t^{-1}]}{(t^n - 1)} \times_{\operatorname{Spec}\mathbb{Z}} X. \tag{3.24}$$

This can equivalently be realised as the kernel sheaf of the map $\mathbb{G}_m \to \mathbb{G}_m$, where $\mathbb{G}_m$ is the multiplicative group scheme represented by $\mathbb{G}_{m,X} := \operatorname{Spec}\mathbb{Z}[t, t^{-1}] \times X$; explicitly we have $\mathbb{G}_m(U) = \Gamma(\mathcal{O}_U, U)^\times$. The corresponding sequence

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{s \mapsto s^n} \mathbb{G}_m \to 0 \tag{3.25}$$

need not be exact in general. This is called the *Kummer sequence*.

Exercise. Suppose n is invertible everywhere on X.

(i) Show that the Kummer sequence is not exact on Zariski sheaves.
(ii) Show that the Kummer sequence is not exact on Zariski presheaves.
(iii) Show that the Kummer sequence is exact in the category of étale sheaves.

Example 3.37 (The Artin-Schreier sequence)**.** Let X be a scheme over a field of characteristic p, and let $\mathbb{G}_a$ be the sheaf on $X_{\text{ét}}$ given by $\mathbb{G}_a(U) = \Gamma(\mathcal{O}_U, U)$. Then we have a sequence

$$0 \to \underline{\mathbb{Z}/p\mathbb{Z}} \to \mathbb{G}_a \xrightarrow{F - \mathrm{Id}} \mathbb{G}_a \to 0, \tag{3.26}$$

where F is the Frobenius map on $\mathbb{G}_a$. This is called the *Artin-Schreier sequence*. As with the Kummer sequence, this is not exact on the right in the categories of Zariski sheaves or presheaves, but it is exact in the category of étale sheaves, essentially because the polynomial $T^p - T$ is separable (exercise!).

Just like with topological spaces, there is a canonical way of producing a sheaf on any site from a given presheaf.

Theorem 3.38. *Let X_E be a site. For any presheaf $\mathcal{P} \in \mathrm{pSh}(X_E)$, there exists a sheaf $\mathcal{P}^a \in \mathrm{Sh}(X_E)$ such that for any sheaf $\mathcal{F} \in \mathrm{Sh}(X_E)$ and morphism of presheaves $\mathcal{P} \to \mathcal{F}$, there exists a unique morphism of sheaves $\mathcal{P}^a \to \mathcal{F}$ so that the following diagram commutes:*

$$\begin{tikzcd} & \mathcal{P}^a \arrow[d, dashed, "\exists!"] \\ \mathcal{P} \arrow[ur, "a"] \arrow[r] & \mathcal{F} \end{tikzcd} \tag{3.27}$$

For a proof, see [Mil80, Thm. II.2.11] or [Sta21, Section 00W1].

Since a is defined by a universal property, the standard argument shows that $\mathcal{F}^a$ is unique up to unique isomorphism. In fact, the construction is functorial.

Definition 3.39. The functor $\mathcal{F} \mapsto \mathcal{F}^a$ is called the **sheafification functor**.

One important fact about sheafification is that it preserves stalks: for any geometric point $\overline{x}$ on X, we have that $(\mathscr{F}^a)_{\overline{x}} = \mathscr{F}_{\overline{x}}$.

Proposition 3.40. *Let X_E be a site.*

(i) *Sheafification is functorial, and the sheafification functor a is left adjoint to the inclusion functor $\mathrm{Sh}(X_E) \hookrightarrow \mathrm{pSh}(X_E)$. Moreover, the functor a is exact.*
(ii) *For a sequence of sheaves $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$, exactness on the left in $\mathrm{Sh}(X_E)$ is equivalent to left exactness as a sequence of presheaves. This is also equivalent to left exactness on the level of sections; if $X_E = X_{\text{ét}}$, then this is also equivalent to left exactness on stalks.*
(iii) *A morphism in $\mathrm{Sh}(X_E)$ is surjective if and only if it is surjective on sections. If $X_E = X_{\text{ét}}$, then this is equivalent to surjectivity on stalks.*
(iv) *Limits in $\mathrm{Sh}(X_E)$ coincide with limits in $\mathrm{pSh}(X_E)$; a colimit in $\mathrm{Sh}(X_E)$ is the sheafification of a colimit in $\mathrm{pSh}(X_E)$*
(v) *The category $\mathrm{Sh}(X_E)$ is abelian and has arbitrary direct sums and products; filtered colimits respect exactness.*

With the axioms from Grothendieck's Tohoku paper, (v) can be rephrased as "$\mathrm{Sh}(X_E)$ satisfies AB5, AB3* (but not AB4*)".[13]

Let $\pi\colon X' \to X$ be a continuous map of sites. Recall that we defined the functor π_* as the restriction of π_p to the category of sheaves. This fails for π^p because $\pi^p\mathscr{F}$ is generally not a sheaf even if $\mathscr{F}$ is. However, we can mend this by sheafifying:

Definition 3.41. Let $\pi\colon X' \to X$ be a continuous map of sites. The functor

$$\pi^*\colon \mathrm{Sh}(X_E) \to \mathrm{Sh}(X'_{E'}), \qquad \mathscr{F} \mapsto \pi^*\mathscr{F} := (\pi^p\mathscr{F})^a \tag{3.28}$$

is called the **pullback along** π, or **inverse image functor**.

By the universal property of sheafification, it is easy to see that (π_*, π^*) form an adjoint pair. Since π^* is not exact in general, neither is π^*. However, by proposition 3.35 π^p *is* for the étale site, and so π^* is as well, being a composition of left exact functors, $\mathrm{Sh} \hookrightarrow \mathrm{pSh} \xrightarrow{\pi^p} \mathrm{pSh} \xrightarrow{a} \mathrm{Sh}$.

For the remainder of the section, we fix a scheme X equipped with the étale topology. Moreover, if $j\colon U \hookrightarrow X$ is an open immersion, we tend to identify U with its image in X. The following proposition tells us how pullbacks and pushforwards interact with stalks.

Proposition 3.42 ([Mil80, Cor. II.3.5])**.** *Let X and X' be schemes.*

[13] AB4* states that an arbitrary product of exact sequences is exact, which is false in general. For some counterexamples, see here.

(i) *For any* $\pi\colon X' \to X$, $\mathscr{F} \in \mathrm{Sh}(X_{\text{ét}})$, *and* $\overline{x}'$ *a geometric point on* X'. *Then* $(\pi^*\mathscr{F})_{\overline{x}'} = \mathscr{F}_{\overline{\pi(x')}}$.

(ii) *If* $j\colon U \to X$ *is an open immersion,* $\mathscr{F} \in \mathrm{Sh}(U_{\text{ét}})$, *and* $\overline{x}$ *a geometric point on* X *such that* $x \in U$, *then* $(j_*\mathscr{F})_{\overline{x}} = \mathscr{F}_{\overline{x}}$.

(iii) *If* $i\colon Z \to X$ *is a closed immersion,* $\mathscr{F} \in \mathrm{Sh}(Z_{\text{ét}})$, *and* $\overline{x}$ *a geometric point on* X, *then*

$$(i_*\mathscr{F})_{\overline{x}} = \begin{cases} \mathscr{F}_{\overline{x}} & \text{if } x \in Z, \\ 0 & \text{if } x \notin Z. \end{cases} \tag{3.29}$$

(iv) *If* $\pi\colon X' \to X$ *is finite and* $\mathscr{F}' \in \mathrm{Sh}(X')$, *then* $(\pi_*\mathscr{F}')_{\overline{x}} = \bigoplus_{x' \mapsto x} (\mathscr{F}'_{\overline{x}})^{d(x')}$, *where* $d(x') = [\kappa(x') : \kappa(\overline{x})]_{\text{sep}}$, *for any geometric point* $\overline{x}$ *of* X.

Definition 3.43. Let $j\colon U \hookrightarrow X$ be an open immersion of schemes, and fix $\mathscr{P} \in \mathrm{pSh}(U_{\text{ét}})$. Define

$$\mathscr{P}_!(V) := \begin{cases} \mathscr{P}(V) & \text{if } \phi(V) \subset U \text{ for an étale morphism } \phi\colon V \to X, \\ 0 & \text{otherwise.} \end{cases} \tag{3.30}$$

This is called "$\mathscr{P}$ **lower shriek**".

If $f\colon \mathscr{P} \to \mathscr{P}'$ is a morphism of presheaves, then we obtain an associated morphism $\mathscr{P}_! \to \mathscr{P}'_!$ by "extending f by 0 outside U"; thus $\mathscr{P} \mapsto \mathscr{P}_!$ is a functor. We can upgrade this to a functor of sheaves by precomposing with the inclusion $\mathrm{Sh} \hookrightarrow \mathrm{pSh}$ and postcomposing with sheafification.

Definition 3.44. Let $U \hookrightarrow X$ be an open immersion of schemes. The **extension by** 0**-functor** $j_!\colon \mathrm{Sh}(U_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$ is given by $\mathscr{F} \mapsto j_!\mathscr{F} := (\mathscr{F}_!)^a$.

It is a straightforward exercise using the universal property of sheafification to show that $(j_!, j^*)$ form an adjoint pair.

Proposition 3.45. *If* $j\colon U \hookrightarrow X$ *is an open immersion,* $\mathscr{F} \in \mathrm{Sh}(U_{\text{ét}})$ *and* $\overline{x}$ *a geometric point of* X, *then* $(j_!\mathscr{F})_{\overline{x}} = \mathscr{F}_{\overline{x}}$ *if* $x \in U$, *and* 0 *otherwise.*

Definition 3.46. Let $i\colon Z \to X$ be a closed immersion and $j\colon U = X \setminus Z \to X$ an open immersion. We define a category $T(X)$ consisting of triples $(\mathscr{F}_1, \mathscr{F}_2, \phi)$ where $\mathscr{F}_1 \in \mathrm{Sh}(Z_{\text{ét}}), \mathscr{F}_2 \in \mathrm{Sh}(U_{\text{ét}})$ and $\phi\colon \mathscr{F}_1 \to i^*j_*\mathscr{F}_2$ is a morphism in $\mathrm{Sh}(Z_{\text{ét}})$.

Theorem 3.47 ([Mil80, Thm. II.3.10])**.** *Fix* $i\colon Z \to X$ *a closed immersion and* $j\colon U = X \setminus Z \to X$ *an open immersion. There is an equivalence of categories*

$$\begin{aligned} \mathrm{Sh}(X_{\text{ét}}) &\to T(X) \\ \mathscr{F} &\mapsto (i^*\mathscr{F}, j^*\mathscr{F}, \phi_{\mathscr{F}}) \\ \psi &\mapsto (i^*\psi, j^*\psi). \end{aligned}$$

Definition 3.48. Let X be a scheme, $Y \hookrightarrow X$ a subscheme and $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$. We say that $\mathcal{F}$ **has support in** Y if $\mathcal{F}_{\bar{x}} = 0$ for every geometric point $\bar{x}$ with image in $X \setminus Y$.

From the previous theorem we deduce the following:

Corollary 3.49 ([Mil80, Cor. II.3.11])**.** *With notation as above, there is an equivalence of categories between* $\mathrm{Sh}(Z_{\text{ét}})$ *and the full subcategory of sheaves on* X *with support in* Z.

Proof (sketch). The main idea here is to show that sheaves with support in Z are equivalent to the subcategory of $T(X)$ given by $(i^*\mathcal{F}, 0, 0)$. □

Definition 3.50. Let $i\colon Z \to X$ be a closed immersion and $j\colon U = X \setminus Z \to X$ an open immersion. Then we have functors

$$\begin{tikzcd} \mathrm{Sh}(Z_{\text{ét}}) \arrow[r, "i_*"] & \mathrm{Sh}(X_{\text{ét}}) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathrm{Sh}(U_{\text{ét}}) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{3.31}$$

which using the equivalence in corollary 3.49 are given explicitly as follows:

$$\begin{aligned} i^* &: \mathcal{F}_1 \leftarrow\!\shortmid (\mathcal{F}_1, \mathcal{F}_2, \phi), & j_! &: (0, \mathcal{F}_2, 0) \leftarrow\!\shortmid \mathcal{F}_2, \\ i_* &: \mathcal{F}_1 \mapsto (\mathcal{F}_1, 0, 0), & j^* &: (\mathcal{F}_1, \mathcal{F}_2, \phi) \mapsto \mathcal{F}_2, \\ i^! &: \ker\phi \leftarrow\!\shortmid (\mathcal{F}_1, \mathcal{F}_2, \phi), & j_* &: (i^* j_* \mathcal{F}_2, \mathcal{F}_2, \mathrm{Id}) \leftarrow\!\shortmid \mathcal{F}_2. \end{aligned} \tag{3.32}$$

Proposition 3.51 ([Mil80, Prop. II.3.14])**.** *Keeping the notation from the previous definition, we have the following:*

(i) *For the top four functors in eq. (3.32), each forms an adjoint pair with the one immediately below.*
(ii) *The functors* i^*, i_*, j^* *and* j_* *are exact.*
(iii) *The functors* i_*, j_* *and* $j_!$ *are fully faithful.*

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