Étale cohomology reading seminar

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3 Étale sheaves

Contents

3.1 Sites and Grothendieck topologies

Speaker: Martin Ortiz Ramirez

Recall from section 1.3 that for the purpose of defining sheaves, we don't need a full topology but rather a notion of open *covers*. In the context of schemes, we do this by viewing an open subset $U \subset X$ as an open immersion $U \hookrightarrow X$, $U \cap V$ as $U \times_X V$, and so on. The axioms required to define a sheaf turn out to be the following:

Definition 3.1. Let C be a category. A **Grothendieck topology** T **on** C consists of collections of distinguished maps $\{U_i \to U\}_{i\in\mathcal{I}}$, *coverings of* U, for each $U \in \mathrm{C}$, satisfying the following axioms:[10]

(i) If $U_i \to U$ and $U_j \to U$ are coverings, then $U_i \times_U U_j \to U$ is also a covering.
(ii) If $\{U_i \to U\}_{i\in\mathcal{I}}$ and $\{U_{ij} \to U_i\}_{j\in\mathcal{J}}$ are coverings of U and U_i respectively, then $\{U_{ij} \to U\}_{(i,j)\in\mathcal{I}\times\mathcal{J}}$ is a covering of U.
(iii) The set consisting of the identity map $\{U \to U\}$ is a covering.

We call the pair (C, T) a **site**.

Example 3.2. If X is any topological space, then the category $\mathrm{U}(X)$ of open subsets where arrows are given by inclusions forms a site, with coverings are

[10] Note: the name "Grothendieck pre-topology" is frequently found in the literature. See this for an explanation of the differences.

given by collections of open inclusions $\{\phi_i \colon U_i \to U\}_{i\in\mathcal{I}}$ such that $\bigcup_i \phi_i(U_i) = U$ ("surjective families").

If X is a scheme, then this is called the **Zariski site**, denoted by X_{Zar}.

Example 3.3. The **small étale site** on a scheme X, $X_{\text{ét}}$, is the category of étale X-schemes $Y \to X$. Note that if $Y \to X$ and $Z \to X$ are étale X-schemes and $Y \to Z$ is a morphism of X-schemes, that is, a morphism of schemes such that the diagram

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Z \arrow[dl] \\ & X & \end{tikzcd} \tag{3.1}$$

commutes, then proposition 2.44 implies that $Y \to Z$ is also étale.

Since every open immersion is étale, there is a natural inclusion of X_{Zar} into $X_{\text{ét}}$.

Recall that a *presheaf* on a category C with values in C′ is a contravariant functor C → C′, and a morphisms of presheaves is simply a natural transformation of functors.

Definition 3.4. A **sheaf on a site** T is a presheaf $\mathcal{F} \colon \mathrm{C} \to \mathrm{C}'$ such that for all coverings $\{U_i \to U\}_{i\in\mathcal{I}}$, the diagram

$$\mathcal{F}(U) \to \prod_{i\in\mathcal{I}} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in\mathcal{I}\times\mathcal{I}} \mathcal{F}(U_i \times_U U_j) \tag{Sh}$$

is an equaliser diagram, which was defined after eq. (1.9) in the introduction. We will refer to this as the *sheaf condition*.

A **morphism of sheaves on** T is a morphism of presheaves, that is, a natural transformation.

As with sheaves on a topological space, we refer to the maps $\mathcal{F}(\phi)$ as *restriction maps*, and the corresponding category is denoted by Sh(T). Unless specified, we assume that C′ = Ab, the category of abelian groups.

Definition 3.5. A sheaf on the small étale site $X_{\text{ét}}$ is called an **étale sheaf**.

Note that every étale sheaf is necessarily also a sheaf for the Zariski site, which is the same as a sheaf in the traditional sense. While it is not always easy to check if a presheaf is a sheaf, the following proposition gives a useful criterion:

Proposition 3.6. *Let $\mathcal{F}$ be a presheaf on the category of étale X-schemes. Then $\mathcal{F}$ is an étale sheaf if and only if it is a sheaf on the Zariski site and for any covering $V \to U$ of affine étale X-schemes, the following is an equaliser diagram:*

$$\mathcal{F}(U) \to \mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_U V). \tag{3.2}$$

In other words, we need only check on affine étale coverings consisting of a single map.

Proof. □

Example 3.7. Given an étale map $U \to X$, define $\mathcal{O}_{X_{\text{ét}}}(U) = \Gamma(U, \mathcal{O}_U)$. This is a Zariski sheaf because it coincides with the structure sheaf when $U \hookrightarrow X$ is an open immersion, and we want to show that it is an étale sheaf by checking the criterion above. If $\operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of X-schemes, then we need to check that

$$A \to B \rightrightarrows B \otimes_A B \tag{3.3}$$

is an equaliser diagram. Here the double arrow corresponds to $b \mapsto b \otimes 1$ and $b \mapsto 1 \otimes b$. Since the category of rings is additive, this is equivalent to exactness of

$$0 \to A \to B \xrightarrow{b \mapsto b \otimes 1 - 1 \otimes b} B \otimes_A B, \tag{3.4}$$

which follows from the fact that $\operatorname{Spec} A \to \operatorname{Spec} B$ is faithfully flat.

Example 3.8. Let Z be an X-scheme, and consider the presheaf $U \mapsto \operatorname{Hom}_X(U, Z)$. It is not difficult to check that this is in fact a sheaf on X_{Zar}, and we claim that it is an étale sheaf. For affine $Z \cong \operatorname{Spec} R$, the exactness of the equaliser diagram

$$Z(A) \to Z(B) \rightrightarrows Z(B \otimes_A B) \tag{3.5}$$

follows from exactness of eq. (3.4), since the associated diagram of rings is

$$\operatorname{Hom}(R, A) \to \operatorname{Hom}(R, B) \rightrightarrows \operatorname{Hom}(R, B \otimes_A B). \tag{3.6}$$

This extends to not necessarily affine Z through a standard patching argument.

For a concrete example, taking

$$Z = \operatorname{Spec} \frac{\mathbb{Z}[t, t^{-1}]}{(t^n - 1)} \times_{\operatorname{Spec} \mathbb{Z}} X$$

we obtain μ_n, which is the usual group scheme with $\mu_n(U) = \ker(\Gamma(U, \mathcal{O}_U) \xrightarrow{s \mapsto s^n} \Gamma(U, \mathcal{O}_U))$ of n-th roots of unity.

Example 3.9. Let X be a quasi-compact scheme, A an abelian group, and let $\underline{A}$ denote the presheaf which sends U to the set of functions $U \to A$ which are constant on each connected component. We recognise this as the sheafification of the constant presheaf $U \mapsto A$.[11] The sheaf $\underline{A}$ is called the **constant sheaf** associated to A.

Example 3.10. Anologously to the Zariski case, we can define a **locally constant sheaf** $\mathcal{F}$ for the étale topology by requiring that for some covering $\{U_i \to U\}_{i \in \mathcal{I}}$, $\mathcal{F}|_{U_i}$ is constant for all $i \in \mathcal{I}$. We will see an example of a locally constant non-constant étale sheaf in the next section.

[11] We have not proved this, but there is a sheafification functor on the category of étale sheaves. See [Sta21, Section 00W1] for more details.

3.2 Étale sheaves over a field

Let G be a group. By a G-module, we mean a module of the associated group ring, $\mathbb{Z}[G]$ which consists of finite formal sums of elements of g, with multiplication given by the group operation. If G is a compact topological group, then we say a G-module M is *discrete* if the stabiliser of each element of M is an open subgroup of G. This is equivalent to endowing M with the discrete topology and requiring the action of G to be continuous.

Example 3.11. If k is a field, then we can consider a *separable closure* k^{sep}, which by definition is the union of all finite separable extensions of k inside a fixed algebraic closure k^{alg}. It is not difficult to show that k^{sep} is a Galois extension, and we let $G \coloneqq \mathrm{Gal}(k^{\mathrm{sep}}/k)$.

G is an example of a *profinite group*, a topological group isomorphic to an inverse limit of finite groups viewed as discrete topological groups: a fundamental result in Galois theory states that $\mathrm{Gal}(k^{\mathrm{sep}}/k) = \varprojlim \mathrm{Gal}(L/k)$, where L runs over finite Galois extensions of k. Moreover, any subextension of k^{sep} is naturally a discrete G-module.

In this section, the goal is to prove the following theorem:

Theorem 3.12 ([Mil80, Thm. II.1.9])**.** *Let k be a field, k^{sep} a fixed separable closure, and $G \coloneqq \mathrm{Gal}(k^{\mathrm{sep}}/k)$. There is an equivalence of categories between the category of étale sheaves on $\mathrm{Spec}\, k$ and the category of discrete G-modules.*

To prove this, it is convenient to introduce the notion of an *étale algebra over k*, which is a finite product of finite separable extensions of k. A ring A is an étale algebra if and only if the map $\mathrm{Spec}\, A \to \mathrm{Spec}\, k$ is étale. Étale k-algebras form a category $\mathrm{Alg}_{\text{ét}}(k)$ with morphisms given by k-algebra maps.

If $\mathscr{F}$ is a presheaf on $X_{\text{ét}}$ where $X = \mathrm{Spec}\, k$, then by composing with the functor Spec we can naturally identify $\mathscr{F}$ with a *covariant* functor $\mathrm{Alg}_{\text{ét}}(k) \to \mathrm{Ab}$.

Lemma 3.13. *With notation as above, a presheaf $\mathscr{F}$ is an étale sheaf if and only if the following two conditions hold:*

(i) $\mathscr{F}(\prod A_i) = \bigoplus \mathscr{F}(A_i)$ *for all finite sets of étale algebras* $\{A_i\}$*;*
(ii) *for all finite Galois extensions L'/L with L/k a finite separable extension, the fixed set of $\mathscr{F}(L')$ under the action of $\mathrm{Gal}(L'/L)$ equals $\mathscr{F}(L)$.*

Explicitly, $\mathrm{Gal}(L'/L)$ acts on $\mathscr{F}(L)$ by $(\sigma, x) \mapsto \mathscr{F}(\sigma)(x)$ for $x \in \mathscr{F}(L)$.

Proof. In light of proposition 3.6, $\mathscr{F}$ is a Zariski sheaf if and only $\mathscr{F}(\prod A_i) = \bigoplus \mathscr{F}(A_i)$ for all étale algebras A_i because any $U \to \mathrm{Spec}\, k$ is discrete. If this holds, then by passing to restrictions we see that $\mathscr{F}$ is étale if and only if for any pair L'/L of finite separable extensions of k, the diagram

$$\mathscr{F}(L) \to \mathscr{F}(L') \rightrightarrows \mathscr{F}(L' \otimes_L L') \tag{3.7}$$

is an equaliser. If L'/L is Galois, it is easy to deduce the equality $\mathscr{F}(L) = \mathscr{F}(L')^{\mathrm{Gal}(L'/L)}$ from eq. (3.7), hence proving necessity.

Conversely, we first prove that $\mathscr{F}(L) = \mathscr{F}(L')^{\mathrm{Gal}(L'/L)}$ is equivalent to exactness of eq. (3.7) for Galois extensions.

($\Leftarrow$) We have natural maps

$$L' \underset{x\mapsto x\otimes 1}{\overset{x\mapsto 1\otimes x}{\rightrightarrows}} L' \otimes_L L' \xrightarrow{\psi_\sigma} L' \tag{3.8}$$

where $\psi_\sigma \colon x \otimes y \mapsto x\sigma(y)$ for fixed $\sigma \in \mathrm{Gal}(L'/L)$. If $z \in \mathscr{F}(L')$ is in the equaliser of eq. (3.7), then $\mathscr{F}(\sigma)(z) = z$, as required.

($\Rightarrow$) If $z \in \mathscr{F}(L')^{\mathrm{Gal}(L'/L)}$, then ψ_σ is an isomorphism, so $\mathscr{F}(\psi_\sigma)$ is injective, which proves the exactness of eq. (3.7).

Finally, to show that exactness of eq. (3.7) for Galois extensions implies exactness for general extensions, consider the diagram

$$\begin{tikzcd} \mathscr{F}(L) \arrow[r] \arrow[d, "\mathrm{Id}"] & \mathscr{F}(L') \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathscr{F}(L' \otimes_L L') \arrow[d] \\ \mathscr{F}(L) \arrow[r] & \mathscr{F}(L'') \arrow[r, shift left] \arrow[r, shift right] & \mathscr{F}(L'' \otimes_L L'') \end{tikzcd} \tag{3.9}$$

where L''/L' is the Galois closure of an arbitrary finite separable extension L' over L. By assumption the bottom line is exact, and $\mathscr{F}(L) \to \mathscr{F}(L'')$ and $\mathscr{F}(L) \to \mathscr{F}(L')$ are easily seen be injective. A standard diagram chase then gives exactness of the top row. $\square$

We now turn to the construction of the functors of theorem 3.12. Suppose $\mathscr{F}$ is an étale sheaf and let $G := \mathrm{Gal}(k^{\mathrm{sep}}/k)$. Let $M_{\mathscr{F}} := \varinjlim \mathscr{F}(k')$ where k' runs over finite separable extensions of k. It is straightforward to check that the images of the inclusions $\mathscr{F}(k \hookrightarrow k')$ assemble to an injective system of abelian groups. Moreover, this is compatible with the action of G on each $\mathscr{F}(k')$, giving rise to an action of G on $M_{\mathscr{F}}$. Thus $M_{\mathscr{F}}$ is a G-module, and it is a good exercise to convince oneself that it is discrete.

Conversely, given $M \in \mathrm{Mod}(G)$, define a presheaf

$$\mathscr{F}_M \colon \mathrm{Alg}_{\text{ét}}(k) \to \mathrm{Ab} \quad \text{by} \quad \mathscr{F}_M(A) = \mathrm{Hom}_{\mathrm{Mod}(G)}(\mathscr{F}(A), M). \tag{3.10}$$

Here $\mathscr{F}(A) := \mathrm{Hom}_{\mathrm{Alg}(k)}(A, k^{\mathrm{sep}})$. By the fundamental theorem of Galois theory, for a finite separable extension k'/k we have $\mathscr{F}(k') \cong G/\mathrm{Gal}(k^{\mathrm{sep}}/k)$ as G-modules. It follows that $F_M(k') \cong M^{\mathrm{Gal}(k^{\mathrm{sep}}/k')}$. Note that $\mathscr{F}_M$ satisfies the criteria in lemma 3.13 for being an étale sheaf:

(i) $\mathscr{F}_M(\prod k_i) = \bigoplus \mathscr{F}_M(k_i)$ for finite collections of separable extensions k_i/k by the standard properties of Hom;

(ii) For k''/k' finite Galois, $\mathscr{F}_M(k'')^{\mathrm{Gal}(k''/k')} = \mathscr{F}_m(k')$ by the discussion above.

Exercise. Show that $M \mapsto F_M$ is fully faithful and essentially surjective.

This proves theorem 3.12. □

3.3 Henselian rings & étale stalks of the structure sheaf

Speaker: Jay Swar

When we first meet sheaves on topological spaces, a fundamental feature is that isomorphisms can be detected on the level of stalks. It is natural to ask whether the same holds for sheaves on sites, and in particular on the small étale site. First we need to extend the notion of points in a way which is compatible with our idea of coverings as distinguished X-schemes.

Definition 3.14. Let X be a scheme. A **geometric point** $\overline{x}$ is a map $\overline{x}\colon \operatorname{Spec} \Omega \to X$, where Ω is some separably closed field.

By definition, a geometric point $\overline{x}$ specifies a point $x \in X$ along with an embedding $\kappa(x) \hookrightarrow \Omega$. Note that for any étale covering U whose image contains x, the diagram

$$\begin{tikzcd} \operatorname{Spec}\Omega \arrow[r] \arrow[dr, "\overline{x}"'] & U \arrow[d] \\ & X \end{tikzcd} \tag{3.11}$$

commutes. Such a U is called an *étale neighbourhood*, and by abuse of notation we write $\overline{x} \in U$. We can now take the injective limit of sections over such étale neighbourhoods, giving the following definition:

Definition 3.15. The **étale stalk** of X at a geometric point $\overline{x}$ is given by

$$\mathcal{O}_{X,\overline{x}} := \varinjlim_{U \ni \overline{x}} \mathcal{O}_U(U). \tag{3.12}$$

Example 3.16. If $X = \operatorname{Spec} k$ for some field k and $\overline{x} : \operatorname{Spec} \Omega \to X$ is a geometric point, then $\mathcal{O}_{X,\overline{x}} \cong \Omega$.

The étale stalk satisfies many of the same properties as the usual stalk:

Proposition 3.17. *Let X be a scheme, $\overline{x}$ a geometric point and let $\kappa(\overline{x})$ denote the residue field of $\mathcal{O}_{X,\overline{x}}$.*

(i) *$\mathcal{O}_{X,\overline{x}}$ is a local ring.*

(ii) $\mathcal{O}_{X,\overline{x}}$ *is Noetherian.*
(iii) $\dim \mathcal{O}_{X,\overline{x}} = \dim \mathcal{O}_{X,x}$, *that is, the Krull dimension of the étale stalk is the same as that of the usual stalk.*
(iv) *every monic coprime factorisation in* $\kappa(\overline{x})$ *lifts to a factorisation in* $\mathcal{O}_{X,\overline{x}}$.
(v) $\kappa(\overline{x})$ *is separably closed.*

The last two properties do not hold for Zariski stalks, but are useful features of étale stalks.

Definition 3.18. A local ring which satisfies (iv) is said to be **Henselian**, and is **strictly Henselian** if it additionally obeys (v).

The Zariski stalks are not Henselian in general, so this is really a feature of the étale stalks. This is ample motivation to study Henselian local rings in general.

Proposition 3.19. *A local ring A with maximal ideal $\mathfrak{m}$ and residue field κ is Henselian if and only if for all $f_1, \ldots, f_n \in R[x_1 \ldots, x_n]$ with $\det\left(\frac{\partial \overline{f}_i}{\partial x_j}\right) \neq 0$, every common root of the reductions $\overline{f}_1, \ldots, \overline{f}_n \in \kappa[x_1, \ldots, x_n]$ lifts to a common root in R.*

This is reminiscent of Newton's method, or Hensel's lemma from which Henselian rings get their name.

Given any local ring, there is a canonical way to construct an associated Henselian ring, its *Henselisation*.

Proposition 3.20 ([Sta21, Lemma 04GN])**.** *Let A be a local ring. There exists a Henselian local ring A^h and $A \to A^h$ a local homomorphism such that for any local homomorphism $\phi\colon A \to B$ where B is a Henselian local ring, there exists a unique local homomorphism $A^h \to B$ such that the diagram commutes:*

$$\begin{tikzcd} A^h \arrow[dr, "\exists!"] & \\ A \arrow[u] \arrow[r] & B \end{tikzcd} \tag{3.13}$$

Recognising this as a universal property, the usual argument shows that A^h is unique up to unique isomorphism.

Proof. The idea here is to define a category consisting of pairs $(S, \mathfrak{q})$, where S is a ring equipped with an étale ring map $S \to A$, and $\mathfrak{q} \subset S$ is a prime lying above $\mathfrak{m}$, the maximal ideal of A. Morphisms in this category are given by A-algebra morphisms $\phi : S \to S'$ such that $\phi^{-1}(\mathfrak{q}') = \mathfrak{q}$. Then we can set $A^h := \varinjlim_{(S,\mathfrak{q})} S$, which exists because colimits exist in the category of rings. One also checks that this is Henselian; details are provided in the link above. □

Definition 3.21. The ring A^h is called the **Henselisation** of A.

With suitable modification to the argument in proposition 3.20, namely by considering instead triples $(S, \mathfrak{q}, \alpha)$ where α is a fixed map $\kappa(\mathfrak{q}) \to \kappa^{\mathrm{sep}}$, we get a strictly Henselian ring A^{sh}, called the **strict Henselisation** of A.

Example 3.22 (Exercise)**.** If $A = \mathbb{Z}_{(p)}$, then the strict Henselisation of A equals the integral closure of $\mathbb{Z}_{(p)}$ in $\mathbb{Z}_p$.

Example 3.23 (Exercise)**.** Let k be algebraically closed, and $A = \mathcal{O}_{\mathbb{A}_k,x}$ where x is the origin, corresponding to the prime ideal $(x_1, \dots, x_n)$. Then $A^h = k[[x_1, \dots, x_n]] \cap k(x_1, \dots, x_n)^{\mathrm{alg}}$.

Exercise ("You should probably google this"). Let X be a variety over an algebraically closed field k, $P \in X$ a non-singular point, and U some Zariski open neighbourhood of P. Then there exists an étale map $\phi \colon U \to \mathbb{A}_k^n$ sending P to 0.

Lemma 3.24. *Let k be algebraically closed, and let X and Y be k-varieties. If $X \to Y$ is an étale map, then the induced map $\mathcal{O}_{Y,f(\bar{x})} \to \mathcal{O}_{X,\bar{x}}$ is an isomorphism.*

Combining this with the previous, we get the following:

Corollary 3.25. *Étale stalks at non-singular points of k-varieties are isomorphic to the ring in example 3.23.*

This is explained by the following proposition:

Proposition 3.26. *Let X be a scheme, and $\bar{x}$ a geometric point of X with underlying point $x \in X$. Then $\mathcal{O}_{X,\bar{x}} \cong (\mathcal{O}_{X,x})^{sh}$.*

Example 3.27 ("Henselisation does not commute with fibre products")**.** Let L be a field of characteristic 0. Then $\mathcal{O}_{X,\bar{x}} = \bar{L}$, which contains more arithmetic information than just $L \cong \mathcal{O}_{X,x}$. On the other hand, if L/k is Galois, then

$$(\mathcal{O}_{X,\bar{x}})^h \otimes_k \bar{k} = \prod_i \bar{k}, \tag{3.14}$$

so in particular is not the strict Henselisation of k.

Theorem 3.28 (Artin approximation)**.** *Let $\{f_i(x_1, \dots, x_n, y_1, \dots, y_n)\}_i \subset k[x_1, \dots, x_n, y_1, \dots, y_n]$ be a collection of polynomials, and let $\hat{y}_i$ be power series in x_i, i.e. $\hat{y}_i \in k[[[]]x_1, \dots, x_n]$, for $i = 1, \dots, n$. If $f_i(x_1, \dots, x_n, \hat{y}_1, \dots, \hat{y}_n) = 0$ for all i, then there is a collection of polynomials $y_1, \dots, y_n \in k[x_1, \dots, x_n]$ such that*

$$f(x_1, \dots, x_n, y_1, \dots, y_n) = 0 \quad \textit{and} \quad y_i \equiv \hat{y}_i \pmod{(x_1, \dots, x_n)^n}, \tag{3.15}$$

for all $i = 1, \dots, n$.

3.4 Stalks of étale sheaves

Speaker: Martin Gallauer

Let X be a locally Noetherian scheme, $\mathcal{F}$ a Zariski sheaf on X, and fix a point $x \in X$. The usual stalk of $\mathcal{F}$ at x can be described as the colimit $\varinjlim_U \mathcal{F}(U)$, where U runs over Zariski covers $U \xrightarrow{\iota} X$ such that

$$\begin{tikzcd} & U \arrow[d, "\iota"] \\ \operatorname{Spec}\kappa(x) \arrow[ur] \arrow[r, "\overline{x}"] & X \end{tikzcd} \tag{3.16}$$

commutes. If we identify U with an open subset of X, then this reduces to the requirement that $x \in U$. However, eq. (3.16) is much more amenable to generalisation in the relative setting.

Definition 3.29. Let X be a scheme, $\mathcal{F}$ a presheaf on $X_{\text{ét}}$, and $\overline{x}\colon \operatorname{Spec}\kappa(\overline{x}) \to X$ a geometric point of X. The **stalk of $\mathcal{F}$ at $\overline{x}$** is the object

$$\mathcal{F}_{\overline{x}} \coloneqq \varinjlim_{(U,\overline{u})} \mathcal{F}(U), \tag{3.17}$$

where U ranges over étale schemes $U \to X$ along with geometric points $\overline{u}$ of U such that the associated diagrams

$$\begin{tikzcd} & U \arrow[d] \\ \operatorname{Spec}\kappa(x) \arrow[ur, "\overline{u}"] \arrow[r, "\overline{x}"] & X \end{tikzcd} \tag{3.18}$$

commute.

A pair $(U,\overline{u})$ is frequently referred to as an *étale neighbourhood of $\overline{x}$*.

Remark. (i) If $\mathcal{F}$ is a presheaf valued in C, where C is the category of abelian groups, rings or modules, then $\mathcal{F}_{\overline{x}}$ is an object of C; this is equivalent to n the statement that $(\cdot)_{\overline{x}}\colon \mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is a map $\operatorname{Sh}(X_{\text{ét}}) \to \mathrm{C}$. In fact, (exercise!) it naturally determines a functor.

(ii) In the situation above, the colimit eq. (3.17) is *filtered*[12], and it follows (exercise!) that $(\cdot)_{\overline{x}}$ is an exact functor.

(iii) The stalk $\mathcal{F}_{\overline{x}}$ only depends on the choice of separable closure $\kappa(\overline{x})$ up to isomorphism.

Proposition 3.30. *A sequence of étale sheaves $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact if and only if for every geometric point $\overline{x}$ of X, the associated sequence $\mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}} \to \mathcal{H}_{\overline{x}}$ is exact.*

[12] i.e. the colimit over a filtered category, see here.

Proof sketch. This is mostly a formal verification. The key point is to reduce to the following statement: If $U \to X$ is étale, $\mathscr{P} \in \mathrm{Sh}(X_{\text{ét}})$, $s \in \mathscr{P}(U)$ and $s_{\overline{x}} = 0$ for all geometric points $\overline{x}$ of X, then $s = 0$. Let's prove this:

Since $s_{\overline{x}} = 0$, by definition there exists some étale neighbourhood $V_{\overline{u}} \to U$ of $\overline{u}$ such that $s|_{V_{\overline{u}}} = 0$.

$$\begin{tikzcd} & V_{\overline{u}} \arrow[d] \\ & U \arrow[d] \\ \operatorname{Spec} \kappa(\overline{x}) \arrow[r] \arrow[ur, "\overline{u}"] \arrow[uur, bend left, "\overline{v}"] & X \end{tikzcd} \tag{3.19}$$

But then the collection $(V_{\overline{u}} \to U)_{\overline{u}}$ is an étale covering, and so $s = 0$ by the sheaf condition. $\square$

Recall that if $\mathscr{F} = \mathscr{O}_X$ and $\overline{x}$ is any geometric point of X with image x, then $\mathscr{O}_{X,\overline{x}} = (\mathscr{O}_{X,x})^{sh}$. These fit into a diagram

$$\begin{tikzcd} \mathscr{O}_{X,x} \arrow[r] \arrow[dr] & \mathscr{O}_{X,x}^{h} \arrow[r] \arrow[d] & \mathscr{O}_{X,x}^{sh} \cong \mathscr{O}_{X,\overline{x}} \\ & \widehat{\mathscr{O}}_{X,x} & \end{tikzcd} \tag{3.20}$$

Another important feature of stalks in the étale topology is that they admit a natural Galois action. More precisely, if $\kappa(\overline{x})/\kappa(x)$ is the separably closed field extension associated to a geometric point $\overline{x}$ and $\mathscr{F}$ an étale sheaf, then $G := \mathrm{Gal}(\kappa(\overline{x})/\kappa(x))$ acts on $\mathscr{F}_{\overline{x}}$ as follows: for any $\sigma \in G$, a triple $(U, \overline{u}, s)$ where $(U, \overline{u})$ is an étale neighbourhood of $\overline{x}$ and $s \in \mathscr{F}(U)$ is sent to the triple $(U, \overline{u} \circ \sigma, s)$.

Exercise. Check that this induces an action on the stalk $\mathscr{F}_{\overline{x}} = \varinjlim_{(U,\overline{u})} \mathscr{F}(U)$.

As a consequence, the functor $(\cdot)_{\overline{x}}$ is actually a functor $\mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Mod}(G)$, the category of G-modules; it is easy to verify that morphisms are automatically G-equivariant.

Exercise. Prove that this is in fact an equivalence of categories, by showing that $(\cdot)_{\overline{x}}$ coincides with the functor of theorem 3.12.

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