Étale cohomology reading seminar

Håvard Damm-Johnsen

2 Étale morphisms

Contents

2.1 Finite and quasi-finite morphisms

Speaker: Andrés Ibánez Núñes

In what follows, all rings are assumed to be Noetherian, and all schemes locally Noetherian, meaning that they can be covered by spectra of Noetherian rings. Readers unhappy with this restriction of generality might find de Jong a more pleasing resource.

We begin by recalling the notion of a finite morphism (cf. [Har77, II.3]).

Definition 2.1. Let $f\colon X \to Y$ be a morphism of schemes. Then f is **finite** if for any affine open $V = \operatorname{Spec} B \subset Y$, the preimage $f^{-1}(V) = \operatorname{Spec} A$ is affine[2] and the induced map $B \to A$ makes A into a finitely generated B-module.

Example 2.2. All closed immersions are finite, because they locally correspond to maps of the underlying rings of the form $A \to A/I$, and A/I is a finitely generated A-module.

It is frequently useful to have a slightly weaker notion of finiteness:

Definition 2.3. A morphism of schemes $f\colon X \to Y$ is **quasi-finite** if it is of finite type[3] and if for any $y \in Y$, the preimage $f^{-1}(y)$ is a discrete topological space.

[2] That is, f is an affine morphism.

[3] i.e. for any $V = \operatorname{Spec} B \subset Y$, $f^{-1}(V)$ has a finite open affine cover $\{U_i = \operatorname{Spec} A_i\}$ such that each A_i is a finitely generated B-algebra.

In particular, this implies that the fibres are finite.

For convenience, we introduce the notion of a *stable class*:

Definition 2.4. Let $\mathscr{P}$ be a family of morphisms of schemes. $\mathscr{P}$ is a **stable class** if the following hold:

(i) $\mathscr{P}$ contains all isomorphisms;
(ii) $\mathscr{P}$ is stable under composition, meaning that if $f\colon X \to Y$ and $g\colon Y \to Z$ are members of $\mathscr{P}$, then so if $g \circ f$;
(iii) $\mathscr{P}$ is stable under base change, meaning that for any $f\colon X \to Y$ in $\mathscr{P}$, if we have a Cartesian square of the form

$$\begin{tikzcd} Z \arrow[r, "f'"] \arrow[d] & S \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{2.1}$$

then the morphism f' is also a member of $\mathscr{P}$.[4]
(iv) $\mathscr{P}$ is local on the target, that is, for every cover $\{V_i\}$ of Y, $f\colon X \to Y$ is in $\mathscr{P}$ if and only if the restrictions $f|_{f^{-1}(V_i)}\colon f^{-1}(V_i) \to V_i$ are in $\mathscr{P}$.

Example 2.5. The class of morphisms of finite type form a stable class, as does the classes of separated, of proper and of affine morphisms. It is an excellent exercise to list all the types of morphisms of schemes you know and decide which of the above conditions they satisfy.

Proposition 2.6. *The collection of all finite morphisms form a stable class, and so does the collection of quasi-finite morphisms.*

Proof. This is more or less routine, and omitted from the talk. Details can be found in [Mil80, Prop. 1.3]. □

Finite and quasi-finite morphisms into the spectrum of a field have a particularly nice interpretation.

Proposition 2.7. *Let k be a field, and $f\colon X \to \operatorname{Spec} k$ a morphism of finite type. Then the following are equivalent:*

(i) *f is finite;*
(ii) *f is quasi-finite;*
(iii) *X is affine, and $X \cong \operatorname{Spec} A$ where A is a finite dimensional k-algebra.*

[4]Normally we would simply say the "the base change $f'\colon X \times_Y S$ is in $\mathscr{P}$", but the category of locally Noetherian schemes is not closed under fibre products, see this SO post.

Proof. $(i) \Rightarrow (iii)$ follows directly from definitions.

$(iii) \Rightarrow (ii)$: Note that A is Artinian[5] because any ideal of A is k-vector space and so a strictly decreasing sequence of ideals (i.e. vector spaces) $I_0 \supset I_1 \supset \ldots$ stabilises in at most $\dim I_0$ steps. The structure theorem for Artinian rings (cf. [AM94, Thm. 8.7]) then implies that $A = \prod_{i=1}^n A_i$ where each A_i is an Artinian local ring. In particular, for each i, $\operatorname{Spec} A_i$ consists of a single point. Now $\operatorname{Spec} A = \prod_{i=1}^n A_i \cong \sqcup_{i=1}^n \operatorname{Spec} A_i$, so f is indeed quasi-finite.

$(ii) \Rightarrow (i)$ If f is quasi-finite, then since X is the preimage of the unique point of $\operatorname{Spec} k$, the underlying topological space of X is finite and discrete, and we can write $X = \sqcup_{i=1}^n \operatorname{Spec} A_i$ where each A_i is a finitely generated k-algebra and a local ring. As before, A_i is Artinian and $\operatorname{Spec} A_i$ consists of a single point, so $X = \operatorname{Spec} \prod_{i=1}^n A_i$ is the spectrum of a finite-dimensional k-algebra, so f is finite. □

The following result explains the name "quasi-finite".

Proposition 2.8. *Finite morphisms are quasi-finite.*

Proof. Let $f : X \to Y$ be finite. We know that f is of finite type, and it remains to show that the fibres are discrete. By definition of the fibre over $y \in Y$, we have a Cartesian diagram

$$\begin{tikzcd} f^{-1}(y) \arrow[r, "f'"] \arrow[d] & \operatorname{Spec} \kappa(y) \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{2.2}$$

where $\kappa(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y$ denotes the residue field of Y at y. The base change f' of f is finite by stability of finiteness under base change, and applying proposition 2.7 it is quasi-finite, hence has discrete fibres. □

It is not difficult to see that the converse is false. For example, exercise 1 of the exercise sheet ([Mil80, Ex. I.1.6b]) shows that Dedekind domains with finitely many primes are quasi-finite but never finite. Another example is the following:

Example 2.9. Fix a ring A, pick $P = a_n T^n + \ldots + a_0 \in A[T]$, set $B = A[T]/(P(T))$ and let $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be the natural map. Then f is finite if and only if B is a finite A-module, which one checks is equivalent to T being integral over A. But this is true if and only if the leading coefficient a_n of P is a unit.

On the other hand, f is quasi-finite if and only if for any $\mathfrak{p} \in \operatorname{Spec} A$, $B \otimes_A \kappa(\mathfrak{p}) \cong \kappa(\mathfrak{p})[T]/(P(T))$ is a finite-dimensional over $\kappa(\mathfrak{p})$. This is equivalent to requiring $P \not\equiv 0 \pmod{\mathfrak{p}}$ for all primes $\mathfrak{p}$ of A, i.e. that $(a_0, \ldots a_n) = A$. This shows that being quasi-finite is weaker than being finite, in general.

[5] Meaning any descending chain of ideals stabilises in finitely many steps.

2.2 Normalisations

Definition 2.10. A scheme X is **normal** if every stalk $\mathcal{O}_{X,x}$ is integrally closed (in its field of fractions)[6].

The notion of being normal seems to have its origins in arithmetic, and one sees that $\operatorname{Spec}\mathbb{Z}[\sqrt{5}]$ is not normal while $\operatorname{Spec}\frac{1}{2}\mathbb{Z}[\sqrt{5}]$ is. One nice property of normal schemes is that every scheme naturally admits a "normalisation":

Proposition 2.11. *Let X be an integral scheme, K the function field of X and let L/K be a field extension. Then there exists a morphism of schemes $f\colon \widetilde{X} \to X$ characterised uniquely by the following properties:*

(i) *$\widetilde{X}$ is normal,*
(ii) *f is affine,*
(iii) *for any open affine set $U \subset X$, $\mathcal{O}_{\widetilde{X}}(f^{-1}(U))$ is the integral closure of $\mathcal{O}_X(U)$ in L.*

Definition 2.12. The scheme $\widetilde{X}$ is called the **normalisation of** X **in** L, or simply the **normalisation of** X if $L = K$.

Normalisations give rise to a large class of finite morphisms:

Proposition 2.13 ([Mil80, Prop. I.1.1], EGA IV.7.8)**.** *Let X be a normal scheme, and $f\colon \widetilde{X} \to X$ the normalisation of X in L. If L/K is separable, or if X is of finite type over a field k, then f is finite.*

Proposition 2.14. *Let X/k be an integral scheme of finite type over a field k, with function field K. Then the normalisation $\widetilde{X} \to X$ of X in K is finite.*

Proof. We may assume X is affine, $X \cong \operatorname{Spec} A$, where A is an integral finite-dimensional k-algebra. By the Noether normalisation theorem ([AM94, Ex. 5.16]), there exists a finite injective homomorphism $k[T_1, \dots, T_n] \to A$, which extends to $k[T_1, \dots, T_n] \to \widetilde{A}$, where $\widetilde{A}$ is the integral closure of A in K. Since $\operatorname{Spec} k[T_1, \dots, T_n]$ is normal and K is a finite extension of $k(T_1, \dots, T_n)$, we have that $k[T_1, \dots, T_n] \to \widetilde{A}$ is finite by proposition 2.13, and so $A \to \widetilde{A}$ is as well. □

Example 2.15. Let k be a field and

$$A = \frac{k[x,y]}{y^2 - x^3 - x^2}, \qquad X = \operatorname{Spec} A, \tag{2.3}$$

a nodal cubic, singular at $x = 0$.

[6] Recall that this means every element of $\operatorname{Frac}(\mathcal{O}_{X,x})$ which is a root of a monic polynomial with coefficients in $\mathcal{O}_{X,x}$ must lie in $\mathcal{O}_{X,x}$.

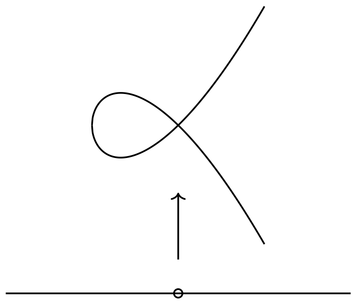


Figure 2: The nodal cubic given by $y^2 = x^3 + x^2$

Consider the map $A \to k[z]$ defined by $x \mapsto z^2 - 1$ and $y \mapsto z^3 - z$ (check by hand that this factors through quotient!). The formal computation $z^2 = y^2/x^2 = (x^3+x^2)/x^2 = x+1$ shows that $k[z]$ is integral over A. In fact, it holds that $\operatorname{Spec} k[z]$ is the normalisation of X.

Removing a single point on X corresponds to localising $k[z]$ at $(z-a)$, and we have a natural morphism $A \to k[z]_{(z-a)}$. The corresponding map of schemes is not finite, but quasi-finite, and factors as an open immersion followed by a finite morphism. This is no coincidence:

Theorem 2.16 (Zariski's main theorem)**.** *Let Y be quasi-compact, and $f : X \to Y$ a separated and quasi-finite morphism. Then f factors as $X \xrightarrow{i} X' \xrightarrow{g} Y$, where i is an open immersion and g a finite morphism.*

Remark. The condition that f be separated is necessary, since a finite morphism is affine, hence separated.

Proof. See [Mil80, Thm. I.1.8]. If we additionally assume that f is projective, then it is possible to deduce this from the Zariski main theorem in Hartshorne's book, [Har77, Cor. III.11.4]. □

This is a different version of Zariski's main theorem from say, the one in [Har77]. For a nice overview of results going by the name "Zariski's main theorem", see [Mum67, Sec. III.9].

We end this section with a useful characterisation of finite morphisms:

Proposition 2.17. *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

(i) *f is finite,*
(ii) *f is proper and quasi-finite,*
(iii) *f is proper and affine.*

Proof. We first prove the equivalence $(i) \Leftrightarrow (ii)$:

$(i) \Rightarrow (ii)$ We already know finite morphisms are quasi-finite, so it remains to prove properness. Recall that being proper means being separated, of finite type and universally closed[7]. Finite morphisms are affine, hence separated ([Har77, Ex. II.5.17b]) and of finite type, so it remains to show that they are universally closed. Since being finite is stable under base change, it suffices to show that f is closed. We reduce further to requiring $f(X)$ to be closed as follows:

If we know that $f(X)$ is closed for all finite morphisms f, then for any closed set $Z \subset X$ we have a closed immersion $Z \to X$, the composition $Z \to X \xrightarrow{f} Y$ is finite, so $f(Z)$ is closed.

In this case we can reduce to the case where Y (and hence X) is affine, since closedness can be checked locally. Then f factors as $X = \operatorname{Spec} A \xrightarrow{u} \operatorname{Spec} B/I \xrightarrow{v} \operatorname{Spec} B = Y$, where u is surjective by the lying-above theorem [AM94, Thm. 5.10], and v is a closed immersion. It follows that $f(X)$ is closed.

$(ii) \Rightarrow (i)$ Since finiteness is local on the target, we can assume that Y is quasi-compact, so by Zariski's main theorem we can factor f as $X \xrightarrow{u} X' \xrightarrow{g} Y$ where u is an open immersion. We claim that u is proper; from this it will follow that u is a closed immersion, so f is a composition of finite morphisms, hence itself finite.

Indeed, let's write u as the composition $X \xrightarrow{(\mathrm{Id}_X, u)} X \times_Y X' \xrightarrow{\mathrm{pr}_2} X'$, where the fibre product is taken over f. Then pr_2 is proper, being the base change of f, and we claim that (Id_X, u) is also proper. Indeed, we have a Cartesian diagram

$$\begin{tikzcd} X \arrow[r, "(\mathrm{Id}_X,u)"] \arrow[d] & X \times_Y X' \arrow[d] \\ X' \arrow[r, "\Delta_g"] & X' \times_Y X' \end{tikzcd} \tag{2.4}$$

which shows that (Id_X, u) is the base change of the diagonal morphism Δ_g, which is a closed immersion since g is separated. Being a closed immersion is stable under base change, so we conclude that (Id_X, u) is indeed a closed immersion, and this proves (i).

$(i) \Rightarrow (iii)$ is now clear using (ii), and the converse follows from finiteness-theorems of proper morphisms, see for example EGA II, 6.7.1. □

2.3 Flat morphisms

Mumford eloquently describes flatness as "a riddle that comes out of algebra, but which technically is the answer to many prayers" [Mum67, Sec. III.10]. One of the solutions he offers is that a flat morphism "preserves linear structure", and in

[7] Any base change of f is closed.

a continuously varying family of schemes we can recognise it by the statement that the dimension of fibres remains constant as the parameter varies.

In the following, we adopt the convention of denoting a short exact sequence of A-modules $0 \to M' \to M \to M'' \to 0$ by Σ, and if N is another A-module, let $\Sigma \otimes_A N$ denote the sequence

$$0 \to M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0. \tag{2.5}$$

Definition 2.18. A map of rings $\phi : A \to B$ is **flat** if for every short exact sequence Σ of A-modules, $\Sigma \otimes_A B$ is exact. A morphism of schemes $f : X \to Y$ is **flat** if for every $x \in X$, the corresponding map of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

Of course, one should check that these notions are compatible if X and Y are affine:

Proposition 2.19. *A morphism of rings $\phi : A \to B$ is flat if and only if the corresponding map* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is flat.*

Proof. This can be rephrased as saying that being flat is a local property, and this is the content of [AM94, Prop. 2.19]. □

Proposition 2.20. *Flat morphisms form a stable class.*

Proof. This is mostly straightforward checking. □

Example 2.21. Open immersions are flat, since each map of stalks is simply the identity.

Another class of morphisms that shows up frequently in scheme theory is the following:

Definition 2.22. A morphism $f : X \to Y$ is **faithfully flat** if it is flat and surjective.

Proposition 2.23. *For a flat morphism of rings $\phi : A \to B$, the following are equivalent:*

(i) *For every A-module M, if $M \neq 0$ then $M \otimes_A B \neq 0$,*
(ii) *for every sequence $\Sigma = 0 \to M' \to M \to M''$ of A-modules, exactness of $\Sigma \otimes_A B$ implies the exactness of Σ.*
(iii) *the associated morphism* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is faithfully flat,*
(iv) *for every maximal ideal $\mathfrak{m} \subset A$, $\phi(\mathfrak{m})B$ is a strict subset of B.*

Checking condition (iv) immediately gives the following:

Corollary 2.24. *A local homomorphism of local rings is faithfully flat.*

Corollary 2.25. *If $f : X \to Y$ is a flat morphism, then $f(X)$ is "closed under generalisation". In other words, if $f(x) \in \overline{\{y\}}$ for some $x \in X$, then $y = f(x')$ for some $x' \in X$.*

Proof. We have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_{X,x} \arrow[r, "\ell"] \arrow[d, hook] & \operatorname{Spec} \mathcal{O}_{Y,f(x)} \arrow[d, hook] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{2.6}$$

We can identify $\operatorname{Spec} \mathcal{O}_{Y,f(x)}$ with the set of generalisations of $f(x)$. By corollary 2.24, the map ℓ is faithfully flat hence surjective, so if $f(x)$ is in the closure of y, then we can find $x' \in \operatorname{Spec} \mathcal{O}_{X,x}$, which we can identify with a generalisation of x. Commutativity of the diagram then implies $f(x') = y$. □

The goal of setting up this machinery is to prove the following important theorem:

Theorem 2.26 ([Mil80],Thm. 1.1.8)**.** *If a morphism $f : X \to Y$ of schemes is flat and locally of finite type, then it is open.*

Proof. To prove this, we will require Chevalley's theorem:

Theorem (Chevalley)**.** *Let $f: X \to Y$ be a morphism of finite type between Noetherian schemes. If $E \subset X$ is constructible*[8]*, then $f(X) \subset Y$ is also constructible.*

We will not prove this here, but one reference is EGA IV, Thm. 1.8.4.

Assume that Y is quasi-compact (hence Noetherian), and that f is of finite type. Flatness being local on the source (exercise!), it suffices to show that $f(X)$ is open, and Chevalley's theorem then implies $f(X)$ is open. The result will then follow from corollary 2.25 and the following lemma:

Lemma 2.27. *Let Y be a Noetherian scheme and let $S \subset Y$ be a subset. Then S is open if and only if S is constructible and stable under generalisation.*

Details to be filled out. □

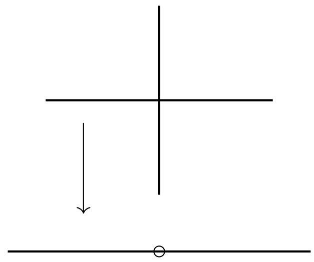
2.4 Unramified morphisms

Speaker: Håvard Damm-Johnsen

[8] Recall that E is *constructible* if it is a finite union of locally closed subsets.

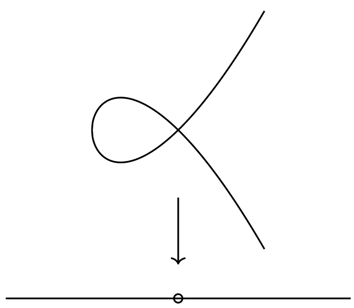
Recall that we are trying to find a nice notion of local isomorphism for schemes. By local, we mean that it should be defined in terms of stalks, and it is desirable to find a notion that holds over arbitrary rings, not just fields.

Example 2.28. Consider the affine scheme $X = \operatorname{Spec} A$ where $A = k[x,y]/(xy)$, regarded as a scheme over $\operatorname{Spec} k[x]$, and we denote by $f : X \to \mathbb{A}^1_k$ the associated morphism. Geometrically, this is a cross along with the projection onto the x-axis.



Heuristically, f is not flat because of the "jump in dimension" of the fibre at 0 compared to the nearby fibres. In formal terms, note that A is a PID, so flatness is equivalent to being torsion-free [LE06, Cor. 1.2.5]. But the localisation $A_{(x,y)}$ viewed as a $k[x]_{(x)}$-module has torsion because $xy = 0$. This demonstrates that flatness should be a necessary condition for being étale. On the other hand, the fibre of (x) is the only place of non-flatness for f, so it should be étale elsewhere.

Example 2.29. Let's return to the nodal cubic in example 2.15, this time with a projection f onto the x-axis.



This corresponds to the natural map $k[x] \to k[x,y]/(y^2 - x^3 - x^2)$, and intuitively f should not be a local isomorphism at the singularity $(0,0)$, because locally there are "four branches" coming out of the point. Readers familiar with Riemann surfaces might recognise this as a ramification point, in the context of which f locally looks like $z \mapsto z^2$ near $(0,0)$.

For[9] a map $g\colon X \to Y$ of Riemann surfaces we have an associated map $\mathcal{M}(Y) \to \mathcal{M}(X)$ of meromorphic function fields, and we can look at the subrings $\mathcal{O}_x \subset \mathcal{M}(X)$, $\mathcal{O}_{g(x)} \subset \mathcal{M}(Y)$ of functions holomorphic at x and $g(x)$, respectively, and g induces a map $\mathcal{O}_{g(x)} \to \mathcal{O}_x$. Here $\mathfrak{m}_{g(x)} \subset \mathcal{O}_{g(x)}$, the ideal of functions vanishing at x, is mapped into the corresponding ideal $\mathfrak{m}_x \subset \mathcal{O}_x$. We see that this is a map of local rings, completely analogous to the map of stalks for f. Identifying the ideal with its image, we see that $\mathfrak{m}_{g(x)}\mathcal{O}_x = \mathfrak{m}_x^{e_x}$, where e_x is the *ramification index*, and corresponds to the "number of branches of g". In this setting, g is said to be *unramified at x* if $e_x = 1$. This transfers almost verbatim to schemes, with the additional requirement of separability.

Definition 2.30. A morphism $f\colon X \to Y$ of schemes locally of finite type is **unramified at** $x \in X$ if the following two conditions hold:

(i) $\mathfrak{m}_{f(x)}$ generates the maximal ideal of $\mathcal{O}_{X,x}$, that is, $\mathfrak{m}_{f(x)}\mathcal{O}_{X,x} = \mathfrak{m}_x$.
(ii) The corresponding field extension $\kappa(x)/\kappa(f(x))$, where $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ and $\kappa(f(x)) := \mathcal{O}_{Y,f(x)}/\mathfrak{m}_{f(x)}$, is separable.

If f is unramified at all $x \in X$, we simply say it is **unramified**.

This definition is sometimes a bit unwieldy; fortunately the following makes computations easier in practice.

Proposition 2.31. *Let $f : X \to Y$ be a morphism locally of finite type. The following are equivalent:*

(i) *f is unramified at x;*
(ii) $(\Omega_{X/Y})_x = 0$;
(iii) *The diagonal morphism $\Delta_{X/Y}$ is an open immersion.*

Proof. $(iii) \Rightarrow (i)$ is somewhat tedious, and we refer the eager reader to [Mil80, Prop. I.3.5].

$(i) \Rightarrow (ii)$: The question is local, so assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Let $\mathfrak{p} = x$, $\mathfrak{q} = f(x)$. Then we have a map $\phi\colon B_\mathfrak{q} \to A_\mathfrak{p}$ which by hypothesis satisfies $\phi(\mathfrak{q})A_\mathfrak{p} = \mathfrak{p}$. It follows that $A_\mathfrak{p} \otimes_{B_\mathfrak{q}} \kappa(\mathfrak{q}) \cong \kappa(\mathfrak{p})$, so we have a Cartesian diagram

$$\begin{tikzcd} \operatorname{Spec}\kappa(\mathfrak{p}) \arrow[r] \arrow[d] & \operatorname{Spec} A_\mathfrak{p} \arrow[d] \\ \operatorname{Spec}\kappa(\mathfrak{q}) \arrow[r] & \operatorname{Spec} B_\mathfrak{q} \end{tikzcd} \tag{2.7}$$

Now [Har77, Prop. II.8.2A] implies that $\Omega_{A_\mathfrak{p}/B_\mathfrak{q}} \otimes_{A_\mathfrak{p}} \kappa(\mathfrak{p}) = \Omega_{\kappa(\mathfrak{p})/\kappa(\mathfrak{q})}$, which vanishes identically by the hypothesis (see [Mum67, p.283] for a hint). Therefore,

[9] Thanks to George for bringing up this analogy!

by Nakayama's lemma, $\Omega_{A_{\mathfrak{p}}/B_{\mathfrak{q}}} = 0$ since it is finitely generated over $A_{\mathfrak{p}}$. Next we have "the first exact sequence"

$$\Omega_{B_{\mathfrak{q}}/B} \otimes B_{\mathfrak{q}} A_{\mathfrak{p}} \to \Omega_{A_{\mathfrak{p}}/B} \to \Omega_{A_{\mathfrak{p}}/B_{\mathfrak{q}}} = 0 \to 0, \tag{2.8}$$

by [Har77, Prop. II.8.3A], which implies that $0 = \Omega_{A_{\mathfrak{p}}/B} = (\Omega_{X/Y})_x$, which proves our claim.

$(ii) \Rightarrow (iii)$ As in the previous part we assume that X and Y are affine, and in this case $\Delta_{X/Y}$ is the map of schemes associated to $m\colon A \otimes_B A \to A$, defined by $m(a \otimes a') = aa'$. Note that m is surjective since $\Delta_{X/Y}$ is a closed immersion, as we are in an affine setting. Note that [Har77, Prop. II.8.1A], $\Omega_{X/Y} = I/I^2$ where $I = \ker m$, so by hypothesis $0 = (\Omega_{X/Y})_x = (I/I^2)_{\Delta(\mathfrak{p})} \cong I_{\Delta(\mathfrak{p})}/I^2_{\Delta(\mathfrak{p})}$. Since f is of finite type, we can apply Nakayama's lemma to deduce that $I_{\Delta(\mathfrak{p})} = 0$. Now by exercise 13.7.E in Vakil's notes, I vanishes in a neighbourhood of U of $\Delta(x)$, so $\Delta|_{\Delta^{-1}(U)}\colon \Delta^{-1}(U) \to U$ is an isomorphism and in particular an open immersion.

□

Returning to the cubic in example 2.29, we compute the sheaf of relative differentials

$$\Omega_{A/k[x]} = \frac{A dx + A dy}{(2y dy - (3x^2 + 2x) dx) A}, \tag{2.9}$$

and one easily checks that the localisation at a prime $\mathfrak{p} \in \operatorname{Spec} A$ is identically 0 if and only if $\mathfrak{p} \neq (x, y)$.

An easy consequence of the above criteria for being unramified is the following:

Proposition 2.32. *Unramified morphisms form a stable class.*

The notion of ramification of schemes also extends the corresponding notion in number theory, as the following example indicates:

Example 2.33. Recall that the prime elements of $\mathbb{Z}[i]$ are given by

(i) primes $p \in \mathbb{Z}$ where $p \equiv 3 \pmod 4$,
(ii) $n + mi$ if $p := n^2 + m^2$ is a prime with $p \equiv 1 \pmod 4$,
(iii) $1 + i$.

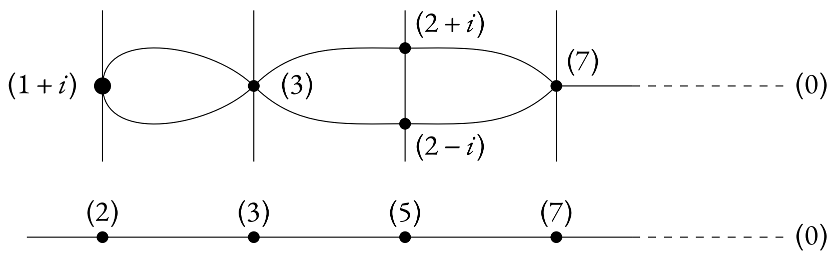
(see e.g. [NS13, Thm. 1.4.]) To study the geometry of $\operatorname{Spec} \mathbb{Z}[i]$, let us consider the fibres under the canonical map f into $\operatorname{Spec} \mathbb{Z}$. Fix a prime $(p) \in \operatorname{Spec} \mathbb{Z}$. Then

$$\operatorname{Spec} \mathbb{Z}[i] \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \kappa(p) = \operatorname{Spec}(\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{F}_p) = \operatorname{Spec} \mathbb{F}_p[i],$$

and consider first the case where $p = 2$. Since $\mathbb{F}_p[i] \cong \mathbb{F}_p[x]/(x^2 + 1)$, this ring has four elements. But via the automorphism $x \mapsto x + 1$, we see that $\mathbb{F}_2[i] \cong \mathbb{F}[x]/x^2$, so the fibre of 2, which consists of only the point $(1+i)$, is a *fat point*, since the fibre is not a field.

Taking $p \equiv 3 \pmod 4$, we claim that the fibre of (p) is a field. Indeed, x^2+1 is irreducible in $\mathbb{F}_p[x]$, hence generates a maximal ideal, so $\mathbb{F}_p[x]/(x^2+1) \cong \mathbb{F}_{p^2}$. On the other hand, if $p \equiv 1 \pmod 4$, then x^2+1 is not irreducible over $\mathbb{F}_p$, but decomposes as the product of two linear factors $P_1(x)$ and $P_2(x)$. Then we have a corresponding decomposition of the fibre, as $\mathbb{F}_p[x]/P_1(x) \times \mathbb{F}_p[x]/P_2(x) \cong \mathbb{F}_p \times \mathbb{F}_p$.

We can draw the picture as follows:



Looking at the local rings, we see that $\mathbb{Z}_{(2)} \to \mathbb{Z}[i]_{(i+1)}$ sends (2) to $(1+i)^2$, so f is ramified at 2.

Exercise. Using the fact that $\mathbb{Z}[\sqrt{d}]$ is ramified precisely at primes dividing the discriminant $4d$, try to draw pictures of $\operatorname{Spec}\mathbb{Z}[\sqrt{d}]$ for some squarefree $d \in \mathbb{Z}$, including composite numbers.

2.5 Étale morphisms

The previous section hopefully convinced you that being flat and unramified are necessary conditions for being a local isomorphism. It turns out that they are also sufficient!

Proposition 2.34 (EGA IV 17.6.3)**.** *Let $f : X \to Y$ be locally of finite type. Suppose $x \in X$ satisfies $\kappa(x) \cong \kappa(f(x))$. Then f is flat at x and unramified at x if and only if the induced map $\widehat{\mathcal{O}}_{Y,f(x)} \to \widehat{\mathcal{O}}_{X,x}$ is an isomorphism.*

Definition 2.35. A morphism $f : X \to Y$ locally of finite type is **étale at** $x \in X$ if it is flat at x and unramified at x. If it is étale at every $x \in X$, we simply say that f **is étale**.

An immediate consequence of proposition 2.20 and proposition 2.32 is the following:

Proposition 2.36. *Étale morphisms form a stable class.*

Example 2.37. The nodal cubic and cross of examples 2.28 and 2.29 respectively, are étale on the complements of the problematic points.

Example 2.38. Fix a Noetherian ring A and $P(x) \in A[x]$. It is natural to ask when the morphism $\operatorname{Spec} A[x]/(P(x)) \to \operatorname{Spec} A$ is étale. It is easy to see that a sufficient condition for flatness is that P be monic, and in general, it turns out that flatness is equivalent to the statement that the ideal of A generated by the coefficients of P is generated by an idempotent.

To be unramified, we recall from Galois theory that a necessary and sufficient condition is that $P(x)$ is separable, that is, has no repeated roots. This is equivalent to the statement that $(P(x), P'(x)) = 1$, where $P'(x)$ is the formal derivative of P, and we can rephrase this as saying that $P'(x) \in (A[x]/P(x))^\times$. Here the map $\operatorname{Spec} A[x]/(P(x)) \to \operatorname{Spec} A$ is a special case of what we call a *standard étale morphism*.

Definition 2.39. Let A be a Noetherian ring, $P(x) \in A[x]$ be a monic polynomial, $B := A[x]/(P(x))$ and fix $b \in B$ such that $P'(x) \in B[b^{-1}]^\times$. A **standard étale morphism** is a morphism of the form $\operatorname{Spec} B[b^{-1}] \to \operatorname{Spec} A$.

The reason for the name is that all étale morphisms locally look like standard étale morphisms.

Theorem 2.40. *Let $f : X \to Y$ be a morphism locally of finite type. Then f is étale at $x \in X$ if and only if there exist affine open neighbourhoods U containing x and V containing $f(x)$ such that $f|_U : U \to V$ is a standard étale morphism.*

Proof. See [Mil80, Thm. 1.3.14], or [Sta21, Section 02GH] for a slightly more modern treatment. □

Corollary 2.41. *A morphism $f : X \to Y$ locally of finite type is étale at $x \in X$ if and only if there exist affine open neighbourhoods $U \cong \operatorname{Spec} R$ containing x and $V \cong \operatorname{Spec} S$ containing $f(x)$ such that*

$$R \cong \frac{S[T_1, \dots, T_n]}{(P_1, \dots, P_n)} \quad \text{and} \quad \det\left(\frac{\partial P_i(T_1, \dots, T_n)}{\partial T_j}\right)_{i,j} \in R^\times \tag{2.10}$$

This is frequently referred to as the "Jacobian criterion" for étale morphisms, and should be seen as an analogue of the implicit function theorem from differential geometry.

Proof. $\Leftarrow$ To show that f is unramified at x, it suffices to show that $(\Omega_{R/S})_x = 0$. By definition,

$$\Omega_{R/S} = \frac{\langle dT_1, \dots, dT_n \rangle_R}{\left\langle \frac{\partial P_i}{\partial T_1} dT_1 + \dots + \frac{\partial P_i}{\partial T_n} dT_n : i = 1, \dots, n \right\rangle_R}, \tag{2.11}$$

and since $\det\left(\frac{\partial P_i(T_1, \dots, T_n)}{\partial T_j}\right)_{i,j} \in R^\times$, the quotient is related to $dT_1, \dots, dT_n$ by the linear transformation corresponding to the Jacobian matrix. Since this is invertible, we are quotienting by everything, and in particular the stalk at x vanishes identically.

Flatness at x follows from an argument similar to (but slightly more involved than) the example above, see [Mum67, Thm. III.10.3'] for more details.

$\Rightarrow$ By theorem 2.40, we can find affine open neighbourhoods $U \cong \operatorname{Spec} R$ and $V \cong \operatorname{Spec} S$ such that $R \cong \left(\frac{S[T]}{P(T)}\right)[b^{-1}]$ for some $b \in \frac{S[T]}{P(T)}$ such that $P'(T) \in R^\times$. Now note that b^{-1} is a zero of the polynomial $bU - 1 \in \frac{S[T]}{P(T)}[U]$, and so

$$R = \frac{S[T,U]}{(P(T), bU - 1)}. \tag{2.12}$$

It remains to check that the corresponding Jacobian matrix is invertible. But

$$\det \begin{pmatrix} \frac{\partial P(T)}{\partial T} & \frac{\partial P(T)}{\partial U} = 0 \\ \frac{\partial (bU-1)}{\partial T} & \frac{\partial (bU-1)}{\partial U} \end{pmatrix} = P'(T) \cdot b, \tag{2.13}$$

which is in $R^\times$ by assumption. This actually proves the slightly stronger statement that we can take $n = 2$. $\square$

An easy consequence of this is the following:

Corollary 2.42. *Let $f : X \to Y$ be étale. Then*

(i) $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,f(x)}$ *for all* $x \in X$;
(ii) *if Y is normal, then X is normal;*
(iii) *if Y is regular, then X is regular.*

Recall from the guiding examples in the beginning of this section that on the complement of a closed set, f was étale. This is no accident; the "problematic points" always form a closed set, as the following proposition shows.

Proposition 2.43. *Let $f : X \to Y$ be locally of finite type. Then the étale locus, meaning the set of points $x \in X$ at which f is étale, is an open set.*

Proof. Evidently the étale locus is the intersection of the set of flat points and the set of unramified points. The flat locus is open by commutative algebra (see EGA IV Thm. 11.3.1 or [MR89, §24]), and the unramified locus is open since it is cut out by the *different ideal sheaf*, as explained in the exercises for this week. $\square$

A useful result for later is the following:

Proposition 2.44. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes such that gf is étale and g is unramified. Then f is étale.*

Proof. We apply the trick of factoring f as $\mathrm{pr}_2 \Gamma_f$ from proposition 2.17. Recall that the graph morphism Γ_f is defined as the base change of the diagonal morphism Δ_g,

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"] & X \times_Z Y \arrow[r, "\mathrm{pr}_2"] \arrow[d] & Y \\ Y \arrow[r, "\Delta_g"] & Y \times_Z Y & \end{tikzcd}$$

and since g is unramified, proposition 2.31 implies that Δ_g is an open immersion, hence étale. Now Γ_f is étale, since étale is stable under base change.

Similarly, the morphism pr_2 arises from the usual Cartesian diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\mathrm{pr}_2"] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "gf"] & Z \end{tikzcd}$$

and since the composition gf is étale, so is its base change pr_2. Since being étale is stable under composition, this proves our result. □

Bibliography

[AM94] M.F. Atiyah and I.G. MacDonald. *Introduction to Commutative Algebra*. Addison-Wesley Series in Mathematics. Avalon Publishing, 1994. 9, 10, 12, 13

[Cho06] Timothy Y. Chow. You could have invented spectral sequences. *Notices of the American Mathematical Society*, 53(1):15–19, 2006. 5

[Har77] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1977. 5, 7, 11, 12, 16, 17, 38, 52

[KS13] Masaki Kashiwara and Pierre Schapira. *Sheaves on Manifolds: With a Short History. «Les Débuts de La Théorie Des Faisceaux». By Christian Houzel*. Springer Science & Business Media, March 2013.

[LE06] Q. Liu and R. Erne. *Algebraic Geometry and Arithmetic Curves*. Oxford Graduate Texts in Mathematics (0-19-961947-6). Oxford University Press, 2006. 15

[Mil80] James S. Milne. *Etale Cohomology (PMS-33)*. Princeton University Press, 1980. 8, 9, 10, 11, 14, 16, 19, 25, 33, 34, 35, 36, 37, 38, 46, 47, 50, 51, 55, 56, 57, 59

[MR89] H. Matsumura and M. Reid. *Commutative Ring Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1989. 20

[Mum67] David Mumford. *Introduction to Algebraic Geometry*. Department of Mathematics, Harvard University, 1967. 11, 12, 16, 20

[NS13] J. Neukirch and N. Schappacher. *Algebraic Number Theory*. Grundlehren Der Mathematischen Wissenschaften. Springer Berlin Heidelberg, 2013. 17

[Ser73] Jean-Pierre Serre. *A Course in Arithmetic*. Graduate Texts in Mathematics. Springer-Verlag, New York, 1973. 54

[Ser95] Jean-Pierre Serre. *Local Fields*. Graduate Texts in Mathematics. Springer New York, 1995. 54

[Ser02] Jean-Pierre Serre. *Galois Cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2002. 44, 45, 54

[Sha13] Igor R. Shafarevich. *Basic Algebraic Geometry 2: Schemes and Complex Manifolds*. Springer-Verlag, Berlin Heidelberg, third edition, 2013. 56

[Sta21] The Stacks project authors. The stacks project. 2021. 19, 25, 28, 32, 34, 54, 55, 59

[Wei94] Charles A Weibel. *An Introduction to Homological Algebra*. Number 38. Cambridge university press, 1994.