Computations with étale cohomology

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Divisors

Let X be a regular, integral and quasi-compact scheme with function field K. Let $g : \mathrm{Spec}(K) \to X$ be the inclusion of the generic point. Denoting by $R(U)$ the rational functions on some $U \to X$ étale, we have $\Gamma(U, g_* \mathbb{G}_{m,K}) = \Gamma(U \times_X \mathrm{spec}(K), \mathbb{G}_m) = R(U)^*$. Hence the maps $\Gamma(U, \mathcal{O}_U^*) \to R(U)^*$ induce an injection of sheaves $\mathbb{G}_{m,X} \to g_* \mathbb{G}_{m,K}$.

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Let X_1 be the set of points on X of codimension 1, so all local rings $\mathcal{O}_{X,x}$ are discrete valuation rings. For $x \in X_1$, denote the inclusion $i_x : x \to X$.

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Proposition

The sheaves D_X and Div_X are isomorphic by sending a rational function to its divisor $(\mathrm{ord}_x(f))_x$.

Computing $H^1(X_{\text{et}}, \mathbb{G}_m)$ (1/3)

We thus have an exact sequence

$$0 \to \mathbb{G}_{m,X} \to g_* \mathbb{G}_{m,K} \to D_X \to 0.$$

This gives rise to a long exact sequence, so to compute $H^r(X_{\text{et}}, \mathbb{G}_{m,X})$ we must compute $H^r(X_{\text{et}}, D_X)$ and $H^r(X_{\text{et}}, g_* \mathbb{G}_{m,K})$. The Leray spectral sequence for $i_x : x \to X$ and the constant sheaf $\mathbb{Z}$ is

$$H^p(X_{\text{et}}, R^q i_{x*} \mathbb{Z}) \implies H^{p+q}(x, \mathbb{Z}).$$

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- We know that $H^0(x, \mathbb{Z}) = \Gamma(x, \mathbb{Z}) = \mathbb{Z}$;
- Further; $H^1(x, \mathbb{Z}) = \text{Hom}_{\text{cts}}(G_x, \mathbb{Z}) = 0$, since continuous homomorphisms factor through a finite subgroup of $G_x = \text{Gal}(k(x)^{\text{sep}}/k(x))$, but $\mathbb{Z}$ has no finite subgroups;
- Lastly, $H^2(x, \mathbb{Z}) = \text{Hom}_{\text{cts}}(G_x, \mathbb{Q}/\mathbb{Z})$ by the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ and the fact that $H^r(G_x, \mathbb{Q}) = 0$ for all $r > 0$ as $\mathbb{Q}$ is an injective abelian group.

Computing $H^1(X_{\text{et}}, \mathbb{G}_m)$ (2/3)

$$H^p(X_{\text{et}}, R^q i_{x*}\mathbb{Z}) \implies H^{p+q}(x, \mathbb{Z})$$

By the same reasoning we can show that $R^1 i_{x*}\mathbb{Z} = 0$. So from the spectral sequence:

- The bottom left entry remains unchanged, so $H^0(X_{\text{et}}, i_{x*}\mathbb{Z}) = \mathbb{Z}$;
- The degree 1 part already has a zero, and must become zero: $H^1(X_{\text{et}}, i_{x*}\mathbb{Z}) = 0$;
- From the filtration in degree 2 we get an inclusion $H^2(X_{\text{et}}, i_{x*}\mathbb{Z}) \hookrightarrow \text{Hom}_{\text{cts}}(G_x, \mathbb{Q}/\mathbb{Z})$.

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Hence by summing over all $x \in X_1$, we obtain that

- $H^0(X_{\text{et}}, D_X) = \bigoplus_{x \in X_1} \mathbb{Z}$
- $H^1(X_{\text{et}}, D_X) = 0$;
- $H^2(X_{\text{et}}, D_X) \hookrightarrow \bigoplus_{x \in X_1} \text{Hom}_{\text{cts}}(G_x, \mathbb{Q}/\mathbb{Z})$.

We will do a similar trick to compute $H^r(X_{\text{et}}, g_*\mathbb{G}_{m,K})$.

Computing $H^1(X_{\text{et}}, \mathbb{G}_m)$ (3/3)

Now we have a Leray spectral sequence of the form

$$H^p(X_{\text{et}}, R^q g_* \mathbb{G}_{m,K}) \implies H^{p+q}(\text{spec}(K), \mathbb{G}_m).$$

Recall that the stalk of $R^1 g_* \mathbb{G}_{m,K}$ at a geometric point $\bar{x}$ of X equals $H^1(K_{\bar{x}}, \mathbb{G}_m) = 0$, where $K_{\bar{x}} = \text{Frac}(\mathcal{O}_{X,\bar{x}})$ and we used Hilbert 90. Since all stalks are zero, it follows that $R^1 g_* \mathbb{G}_{m,K} = 0$. Using the same reasoning as before, we find

- $H^0(X_{\text{et}}, g_* \mathbb{G}_{m,K}) = H^0(K, \mathbb{G}_{m,K}) = K^*$;
- $H^1(X_{\text{et}}, g_* \mathbb{G}_{m,K}) = 0$;
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Then the long exact sequence becomes

$$0 \to \Gamma(X_{et}, \mathcal{O}_X^*) \to K^* \to \bigoplus_{x \in X_1} \mathbb{Z} \to H^1(X_{et}, \mathbb{G}_m) \to 0$$

$$0 \to H^2(X_{et}, \mathbb{G}_m) \to H^2(K, \mathbb{G}_{m,K})$$

We conclude that

$$H^1(X_{et}, \mathbb{G}_m) = \text{divisors}/\text{principal divisors} = \text{Pic}(X).$$

Stronger results (1/2)

Suppose X has dimension 1 and that for all $x \in X_1$, the field $k(x)$ is perfect. Then $K_{\bar{x}}$ is the fraction field of a Henselian DVR with algebraically closed residue field. Group cohomology shows that for such fields, $H^2(K_{\bar{x}}, \mathbb{G}_m) = 0$. These are the stalks of $R^2 g_* \mathbb{G}_{m,K}$, and so this sheaf vanishes.

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Since X has dimension 1, the points $x \in X_1$ are closed, and so i_{x*} is exact, and so $R^q i_{x*} = 0$ for all $q > 0$. So as before, from the spectral sequences we get isomorphisms $H^q(X_{\text{et}}, i_{x*}\mathbb{Z}) = H^q(x, \mathbb{Z})$ for all $q > 0$ and $H^2(X_{\text{et}}, g_* \mathbb{G}_{m,K}) = H^2(\text{spec}(K), \mathbb{G}_m)$.

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We thus obtain the exact sequence

$$0 \to H^2(X_{\text{et}}, \mathbb{G}_m) \to H^2(K, \mathbb{G}_{m,K}) \to \bigoplus_{x \in X_1} \text{Hom}_{\text{cts}}(G_x, \mathbb{Q}/\mathbb{Z}) \to$$
$$\to H^3(X_{\text{et}}, \mathbb{G}_m) \to H^3(K, \mathbb{G}_{m,K})$$

Note that we cannot yet extend the exact sequence further.

Stronger results (2/2)

If X is in addition excellent (e.g. $\mathrm{char}(K) = 0$) then one can show that $K_{\bar{x}}$ is a C_1-field, i.e. quasi-algebraically closed, i.e. every non-constant homogeneous polynomial has a non-trivial zero provided the number of its variables is greater than its degree.

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$$\to H^r(X_{\mathrm{et}}, \mathbb{G}_m) \to H^r(K, \mathbb{G}_{m,K}) \to \bigoplus_{x \in X_1} H^{r-1}(k(x), \mathbb{Q}/\mathbb{Z}) \to H^{r+1}(X_{\mathrm{et}}, \mathbb{G}_m) \to$$

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If X is a smooth algebraic curve over an algebraically closed field k, then K must be C_1, and so $H^r(K, \mathbb{G}_m) = 0$ for all $r > 0$. Since k is algebraically closed, we also have $H^r(k(x), \mathbb{Q}/\mathbb{Z}) = 0$ for all $r \geq 1$. Hence the above exact sequence shows that for all $r > 1$,

$$H^r(X, \mathbb{G}_m) = 0.$$

Intermezzo on changing sites (1/2)

Proposition

Let C/X be a subcategory of C'/X and let $f : (C'/X)_E \to (C/X)_E$ be the morphism of sites defined by the inclusion functor $C \to C'$. Then

$$H^i(X, f_* F') \to H^i(X, F') \quad \text{and} \quad H^i(X, F) \to H^i(X, f^* F)$$

are isomorphisms for all $i \geq 0$ and all sheaves F' on $(C'/X)_E$ and F on $(C/X)_E$. Thus, cohomology on the small and big E-sites is the same.

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Proposition

Now suppose that $E_1 \supset E_2$ are nice classes of morphisms and that $C_1 \supset C_2$ are categories. Let $f : (C_1/X)_{E_1} \to (C_2/X)_{E_2}$ be as above. Suppose that for every covering in the E_1-topology there exists a covering in the E_2-topology that refines it. Then $H^i(X_{E_2}, f_* F) \cong H^i(X_{E_1}, F)$ for any sheaf F on X_{E_1}.

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We can use this to restrict from all étale morphisms to the sites (ét), and even to the class of all separated étale morphisms or affine étale morphisms. Also, we can restrict from all smooth morphisms to (ét), etc.

Intermezzo on changing sites (2/2)

Proposition

Suppose $(C/X)_E$ is a site where every covering has a finite subcovering. Let E_f be the class of finite coverings in E. Then the categories of (pre)sheaves on X_{E_f} and X_E are canonically equivalent, so also cohomology agrees. In particular, $X_{\text{et},f}$ gives the same cohomology as the small and big étale sites.

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For a comparision between flat cohomology and Zariski cohomology for quasi-coherent modules, see Milne Prop. 3.7.

Theorem

If G is a smooth, quasi-projective, commutative group scheme over X, then the canonical maps $H^i(X_{\text{et}}, G) \to H^i(X_{\text{fl}}, G)$ are isomorphisms.

The proof is long and technical, see Milne Thm. 3.9. We will instead focus on étale cohomology versus complex cohomology.

Étale versus complex (1/5)

Let $H^i(X(\mathbb{C}), -)$ denote the classical complex cohomology theory.

Theorem

Let X be a smooth scheme over $\mathbb{C}$. Then for any *finite* abelian group M, we have that $H^i(X(\mathbb{C}), M) \cong H^i(X_{\text{et}}, M)$ for all $i \geq 0$.

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For $i = 0$, this just says that X and $X(\mathbb{C})$ have the same number of connected components. We reason as follows. Suppose X is a connected non-singular curve. By adding a finite number of points, we may assume that X is projective. Suppose X is disconnected in the complex topology, say $X = X_1 \sqcup X_2$.

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By using Riemann-Roch, we can find a non-constant rational function f on X with poles only on X_1. But then f is holomorphic on the compact space X_2, hence locally constant. But then for some $a \in \mathbb{C}$, the complex function $f(z) - a$ has infinitely many zeroes. But then $f(z) = a$ everywhere, a contradiction. The general case can be deduced from this by induction on the dimension. (Shafarevich, Basic Algebraic Geometry, VII.2).

Étale versus complex (2/5)

For $i = 1$, one can show that both H^1-groups are in bijection with Galois coverings with automorphism group M of $X(\mathbb{C})$ and X respectively, see Milne Lecture Notes Prop 11.1 and Example 11.3.

Riemann Existence Theorem

Let X be locally of finite type over $\mathbb{C}$ and let X^{an} be the associated complex analytic space. Then mapping $Y \to Y^{\mathrm{an}}$ defines an equivalence of categories of finite étale coverings Y/X and similar coverings on X^{an}.

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- The same functor on finite étale coverings is fully faithful. Namely, to give a map $Y \to Y'$ over X is to give a section $Y \to Y \times_X Y'$. For étale morphisms, this is to give an isomorphism $\Gamma \to X$ for some connected component Γ of $Y \times_X Y'$. We know these agree.

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- For essentially surjective, reduce to the affine case and resolve all singularities. Then one can show it by hand (SGA 1, XII 5.3).

Étale versus complex (3/5)

We may thus assume that $i > 1$ from now on. Let X_{cx} be the site on X^{an} with morphisms all local isomorphisms of complex analytic spaces. Since for any open U, the map $U \hookrightarrow X(\mathbb{C})$ is a local isomorphism, we obtain a morphism of sites $X_{\mathrm{cx}} \to X(\mathbb{C})$. Since every complex étale cover can be refined to an open cover (inverse function theorem), as in the intermezzo, we find that $H^i(X_{\mathrm{cx}}, M) \cong H^i(X(\mathbb{C}), M)$.

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Since for $U \to X$ étale, the map $U^{\mathrm{an}} \to X^{\mathrm{an}}$ is a local isomorphism, we also obtain a morphism of sites $f : X_{\mathrm{cx}} \to X_{\mathrm{et}}$. There is a Leray spectral sequence

$$H^i(X_{\mathrm{et}}, R^j f_* F) \implies H^{i+j}(X_{\mathrm{cx}}, F).$$

We will show that $R^j f_* F = 0$ for all $j > 0$. Then the spectral sequence immediately degenerates and gives us our desired result.

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We will show that $R^j f_* F = 0$ for all $j > 0$. Then the spectral sequence immediately degenerates and gives us our desired result. Recall $R^j f_* F$ is the sheafification of $U \mapsto H^j(U_{cx}, F)$. We prove the following.

Lemma

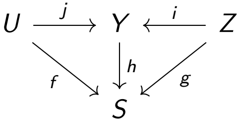
Let U be a connected nonsingular variety and F a locally constant sheaf on U_{cx} with finite stalks. Then for any $t \in H^j(U_{cx}, F)$ where $j > 0$, there exists an étale covering $U_i \to U$ such that t vanishes in each $H^j(U_{i,cx}, F)$.

Étale versus complex (4/5)

Definition

An *elementary fibration* is a regular map of varieties $f : U \to S$ that fits into a commutative diagram as below, where

- j is an open immersion with image dense in every fibre of h, and $Y = i(Z) \sqcup j(U)$;
- h is smooth and projective with geometrically irreducible fibers of dimension 1;
- g is finite and étale with non-empty fibers.



Étale versus complex (4/5)

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$$\begin{array}{ccccc} U & \xrightarrow{j} & Y & \xleftarrow{i} & Z \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle g} & \\ & & S & & \end{array}$$

Lemma

Let X be a non-singular variety over $k = \bar{k}$. For any $x \in X$ closed, there is an elementary fibration $U \to S$ with U an open neighbourhood of x and S non-singular.

Milne is very vague about this, so we will just assume it.

Étale versus complex (5/5)

Lemma

Let U be a connected nonsingular variety and F a locally constant sheaf on U_{cx} with finite stalks. Then for any $t \in H^j(U_{\mathrm{cx}}, F)$ where $j > 0$, there exists an étale covering $U_i \to U$ such that t vanishes in each $H^j(U_{i,\mathrm{cx}}, F)$.

Proof: Because the statement is local for the étale topology, we may assume that $F = M$ is constant and that U admits an elementary fibration $f : U \to S$.

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Proof: Because the statement is local for the étale topology, we may assume that $F = M$ is constant and that U admits an elementary fibration $f : U \to S$. By *Cohomological Purity* (Milne VI.5.1), the fact that f is proper and *Smooth Specialisation of Cohomology Groups* (Milne VI.4.2), one can show that $f_* F$ is also a constant sheaf, $R^1 f_* F$ is a locally constant torsion sheaf with finite fibers and $R^i f_* F = 0$ for $i > 1$. Using those theorems requires that F is torsion.

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$$H^i(S_{\text{cx}}, R^j f_* F) \implies H^{i+j}(U_{\text{cx}}, F)$$

thus only has two non-zero rows. It thus induces a long exact sequence

$$\ldots \to H^i(S_{\text{cx}}, f_* F) \to H^i(U_{\text{cx}}, F) \to H^{i-1}(S_{\text{cx}}, R^1 f_* F) \to \ldots$$

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Use induction on $\dim(U)$ and apply the result of the lemma to S with $\dim(S) < \dim(U)$. We find $V \to S$ étale such that for any $t' \in H^{i-1}(S_{cx}, R^1 f_* F)$ and $t \in H^i(S_{cx}, f_* F)$, their restriction to V is zero. Now consider $V \times U \to U$ to complete the proof.

Wrapping up

Theorem

Let X be a smooth scheme over $\mathbb{C}$. Then for any *finite* abelian group M, we have that $H^i(X(\mathbb{C}), M) \cong H^i(X_{\text{et}}, M)$ for all $i \geq 0$.

Question: What can we say about more general abelian groups M?
Answer: The theorem need not necessarily be true. For example, let X be an elliptic curve over $\mathbb{C}$. Then $H^1(X(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}^2$, whereas

$$H^1(X_{\text{et}}, \mathbb{Z}) = \text{Hom}_{\text{cts}}(\pi_1(X), \mathbb{Z}) = 0.$$

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Recall that the functor F : fin. et.$/X \to$ Set sending (Y, π) to $\text{Hom}_X(\bar{x}, Y)$ for some fixed $\bar{x} \in X$ is pro-representable, i.e. there exists a projective system $\tilde{X} = (X_i)_{i \in I}$ of finite étale coverings such that $F(Y) = \varinjlim \text{Hom}_X(X_i, Y)$, with each $X_i \to X$ a Galois covering. Then

$$\pi_1(X, \bar{x}) := \varprojlim \text{Aut}_X(X_i).$$

This is a profinite group, and thus has no non-trivial continuous homomorphisms to $\mathbb{Z}$. Then $H^1(X_{\text{et}}, \mathbb{Z}) = \text{Hom}_{\text{cts}}(\pi_1(X), \mathbb{Z})$ follows from the Riemann Existence Theorem and the analogous result for $X(\mathbb{C})$.

Fin

Thanks for listening!

Figure: Étale is a mountain of Savoie and Haute-Savoie, France. It lies in the Aravis Range of the French Prealps and reaches 2,484 metres above sea level.