$f \in k[X_0, \ldots, X_{n+1}]$

$X(k) \subseteq \mathbb{P}^{n+1}(k)$ zero set of f

$\left(\frac{\partial f}{\partial X_0}, \ldots, \frac{\partial f}{\partial X_{n+1}}\right) \neq 0$ on $X(k)$

$k = \mathbb{C}$ $\mathbb{P}^{n+1}(\mathbb{C}) = \mathbb{C}\mathbb{P}^{n+1} \supseteq X(\mathbb{C})$

complex mfd of dim n

(of real dim $2n$)

<u>Invariants</u> : cohomology grps

$H^i(X(\mathbb{C}); \mathbb{Q})$ ~>

Betti #'s $b_i(X) := \dim_{\mathbb{Q}} H^i(X(\mathbb{C}); \mathbb{Q})$

$0 \leq i \leq 2n$

$\chi(X) := \sum (-1)^i b_i$

(htpy groups, Hodge groups, ...)

Examples (i) $\deg(f) = 1 \Rightarrow X(\mathbb{C}) = \mathbb{CP}^n$

$b_i : \underset{0}{1}, 0, 1, \ldots, 0, \underset{2n}{1}, \quad \chi = n+1$

(ii) $n = 1$, $\deg(f) = d \Rightarrow$

$X(\mathbb{C})$ Riemann surface of genus

$$g = \frac{(d-1)(d-2)}{2}$$

g holes

$b_i : 1, 2g, 1$

$\chi : 2-2g$

$k = \mathbb{F}_q$ $\quad q$ prime power

$X(k)$ finite set

$X(\mathbb{F}_{q^r})$ $\quad$ "

Invariants: $\# X(\mathbb{F}_{q^r})$

$=: N_r(X)$

Examples: (i) $X = \mathbb{P}^n \Rightarrow$

$$\#\mathbb{P}^n(\mathbb{F}_{q^r}) = q^{r(n+1)} - 1 \,/\, q^r - 1$$

$$= q^{rn} + q^{r(n-1)} + \dots + 1$$

(ii) X ell. curve $(g = 1)$

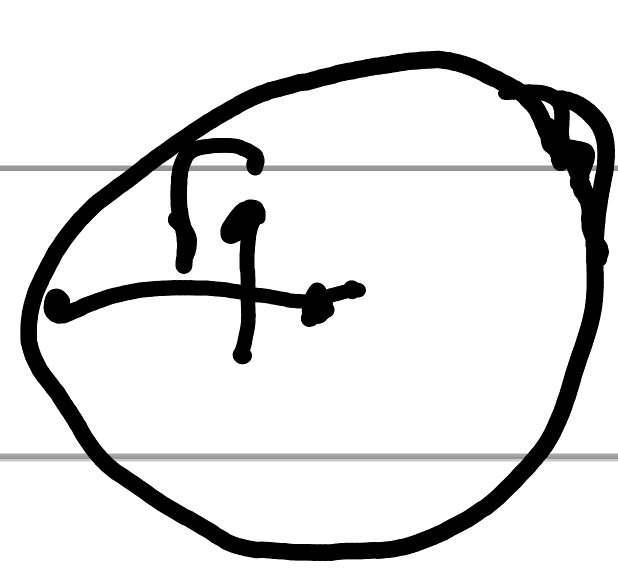
Hasse's Theorem:

$$\left| \#X(\mathbb{F}_{q^r}) - q^r - 1 \right| \leq 2 \cdot q^{r/2}$$

X genus g curve

Theorem (Weil) $\exists$ alg. integers $a_1, \dots, a_{2g}$ s.t.

- $\forall r \geq 1 \quad \#X(\mathbb{F}_{q^r}) = q^r + 1 - (a_1^r + \dots + a_{2g}^r)$
- a_i q-Weil #'s of weight 1 (i.e. $|a_i| = q^{1/2}$)



e.g.

$3 \pm 2i\sqrt{2}$

17-Weil #

$k = \mathbb{Q}$ $\quad f \in \mathbb{Z}[X_0, \dots, X_{n+1}]$ primitive

$\quad \bar{f} \in \mathbb{F}_q[X_0, \dots, X_{n+1}]$ "irr. + smooth"

$\mathcal{X}/\mathbb{Z}$

$$\begin{tikzcd} & \mathcal{X}/\mathbb{Z} \arrow[dl, squiggly] \arrow[dr, squiggly] & \\ \mathcal{X}(\mathbb{C})/\mathbb{C} \arrow[rr, leftrightarrow, squiggly, "\text{relation}"'] & & \mathcal{X}(\mathbb{F}_q)/\mathbb{F}_q \end{tikzcd}$$

$\rightsquigarrow b_i, \chi$ $\qquad$ <u>Weil conjectures</u> $\rightsquigarrow N_r$

geometry/topology $\qquad\qquad$ arithmetic

<u>Definition</u> $X/\mathbb{F}_q$

$$\zeta(X,T) := \exp\left(\sum_{r \geq 1} \frac{N_r(X)}{r} T^r \right)$$

$\in \mathbb{Q}[[T]]$ $\quad$ Zeta fct of $X/\mathbb{F}_q$

<u>Remark</u> $\displaystyle \frac{d}{dT} \log \zeta(X,T) = \sum_{r \geq 0} N_{r+1}(X) T^r$

"generating fct for $(N_r(X))$"

Examples (i) $X = \mathbb{P}^n$

$$\zeta(\mathbb{P}^n, T) = \exp\Big(\sum_{r \geq 1} \frac{1 + q^r + \dots + q^{rn}}{r} T^r\Big)$$

$$= \exp\Big(\sum_{r\geq 1} \frac{1}{r} T^r\Big) \cdot \dots \cdot \exp\Big(\sum_{r \geq 1} \frac{q^{rn}}{r} T^r\Big)$$

$$= \frac{1}{(1-T)(1-qT)\cdots(1-q^n T)}$$

(ii) X curve of genus g $\underset{\text{Thm}}{\overset{\text{Weil's}}{\Longrightarrow}}$

$$\zeta(X,T) = \frac{(1-a_1T)\cdots(1-a_{2g}T)}{(1-T)(1-qT)}$$

Rmk In both cases: rational fct, related to Betti #'s of corresponding varieties/$\mathbb{C}$.

Theorem (Weil conjectures) $X/\mathbb{F}_q$ sm. proj of dim n.

(1) $$\zeta(X,T) = \frac{Q_1 \cdots Q_{2n-1}}{Q_0 \cdots Q_{2n}}, \quad Q_i(T) \in \mathbb{Z}[T],$$

$$Q_i(T) = \prod_{j=1}^{b_i}(1-a_{ij}T)$$

(2) $$\zeta\left(X, \frac{1}{q^n T}\right) = \pm q^{\frac{n\chi}{2}} T^{\chi} \zeta(X,T)$$

$$\chi = \sum (-1)^i b_i$$

(3) a_{ij} are q-Weil #'s of weight i

(4) If X comes from $\mathcal{X}/\mathbb{Z}$ as above then $b_i = b_i(\mathcal{X}(\mathbb{C}))$

Remark Pf of "all" of these used étale cohomology (Dwork (p-adic), Grothendieck, Artin, Deligne)

Remark "Amazing" in many ways.

- Rationality of Z-fct $\Rightarrow$ $(N_r)_{r \geq 1}$ determined by finitely many
- applications to char. 0

Exercise Compute Betti #'s of complex Grassmannians $G(\ell, d)$, $\ell \leq d$ by counting pts / finite fields

b_i = # paths