Spectral sequences

Lukas Kofler

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Derived functors

Let $\mathcal{A}$ be an abelian category. An object $I \in \mathcal{A}$ is *injective* if $A \mapsto \mathrm{Hom}_{\mathcal{A}}(A, I)$ is an exact functor. We say that $\mathcal{A}$ has *enough injectives* if every $A \in \mathcal{A}$ injects into an injective object of $\mathcal{A}$. This lets us define injective resolutions $A \to I^\bullet$.

If $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor between abelian categories, we can define the *right derived functors* $R^n F$ of F. They satisfy $R^0 F = F$, $R^n F(I) = 0$ if I is injective and $n > 0$. They take short exact sequences $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$ to long exact sequences

$$\ldots \to R^n F(A) \to R^n F(A'') \xrightarrow{\delta} R^{n+1} F(A') \to R^{n+1} F(A) \to \ldots$$

Any category of sheaves on a site has enough injectives.

Now we can define the right derived functors of any left exact functor from $\mathbf{S}(X_E)$ into an abelian category.

- The global sections functor $\Gamma(X,-) : \mathbf{S}(X_E) \to \mathbf{Ab}$, $\mathscr{F} \mapsto \mathscr{F}(X)$. Its right derived functors are $R^i\Gamma(X,-) = H^n(X,-)$.
- The inclusion functor $i : \mathbf{S}(X_E) \to \mathbf{P}(X_E)$. Its right derived functors are $\underline{H}^i(F)$.
- For a fixed sheaf $\mathscr{F}_0$ on X_E, the functor $\mathscr{F} \mapsto \mathrm{Hom}_{\mathbf{S}}(\mathscr{F}_0, \mathscr{F})$ is left exact, with right derived functors $\mathrm{Ext}^i_{\mathbf{S}}(\mathscr{F}_0, -)$.
- For a fixed sheaf $\mathscr{F}_0$ on X_E, $\underline{\mathrm{Hom}}(F_0, F)$ is the sheaf $U \mapsto \mathrm{Hom}(\mathscr{F}_0|U, \mathscr{F}|U)$. This gives a left exact functor $\mathbf{S}(X_E) \to \mathbf{S}(X_E)$ with right derived functors $\underline{\mathrm{Ext}}^i(\mathscr{F}_0, \mathscr{F})$.
- For a continuous morphism between sites $\pi : X'_{E'} \to X_E$, the pushforward π_* is left exact. Its right derived functors are denoted $R^i\pi_*$. For a sheaf $\mathscr{F}$ on $X'_{E'}$, the $R^i\pi_*\mathscr{F}$ are called the *higher direct images* of $\mathscr{F}$.

Note: left exactness is not necessary but guarantees $R^0F = F$.

Fix an abelian category $\mathcal{A}$, say R-$\mathbf{mod}$ or $\mathbf{S}(X_E)$.
A (cohomological) *double complex* in $\mathcal{A}$ is a family $\{E_0^{p,q}\}$ of objects in $\mathcal{A}$, together with maps

$$d_h : E_0^{p,q} \to E_0^{p+1,q} \quad \text{and} \quad d_v : E_0^{p,q} \to E_0^{p,q+1},$$

such that $(d_h)^2 = 0$, $(d_v)^2 = 0$, and $d_h d_v + d_h d_v = 0$.
We will only deal with first quadrant double complexes, i.e. $E_0^{p,q} = 0$ unless $p, q \geqslant 0$.
Examples: resolution of a complex; double complex induced by a filtered complex.
From a double complex $E^{\bullet\bullet}$ we can construct its *total complex* $E^{\bullet}$ defined by $E^k = \bigoplus_i E^{i,k-1}$ with differential $d = d_h + d_v$. We want to compute the cohomology of the total complex.
As a first step, we can compute the "vertical" cohomology of the double complex, by considering only the action of d_v on $E^{\bullet\bullet}$ and forgetting about d_h. Setting

$$E_1^{p,q} = \ker d_v^{p,q} / \operatorname{im} d_v^{p,q-1},$$

we get another double complex. Applying d_v to it again would have no effect.
The horizontal differential d_h descends to a differential d_1 on $E_1^{\bullet\bullet}$, so we can apply it and take cohomology again. We set

$$E_2^{p,q} = \ker d_h^{p,q} / \operatorname{im} d_h^{p-1,q}.$$

Spectral sequences: motivation

Is this all we need to do to calculate $H^n(E^\bullet)$? Not quite.

Special case: if only columns p and $p+1$ of $E^{\bullet\bullet}$ are non-zero, then we have computed $H^n(E^\bullet)$ up to extension: there is a short exact sequence

$$0 \to E_2^{pq} \to H^{p+q}(E^\bullet) \to E_2^{p+1,q-1} \to 0.$$

In general, there is another natural map $d_2 : E_2^{pq} \to E_2^{p+2,q-1}$:

Take $x \in E_2^{pq}$ and lift it to $x' \in E_1^{pq}$. Then $d_1(x') = 0$ in $E_1^{p+1,q}$. This means that for a lift of x' to $x'' \in E_0^{pq}$, $d_h(x)$ is in the image of d_v, say $d_h(x) = d_v(y)$, for $y \in E_0^{p+1,q-1}$. $d_h(y)$ lies in $E_0^{p+2,q-1}$ and $d_v d_h(y) = -d_h d_v(y) = -(d_h)^2(x) = 0$, so $d_h(y)$ lies in the kernel of d_v and therefore gives an element of $E_1^{p+2,q-1}$. But $d_1 d_h(y) = 0$, so we get an element of $E_2^{p+2,q-1}$.

This map is well-defined and a differential.

It doesn't stop here, there are more differentials d_r.

Given the pattern we're seeing, it's time to make a definition.

A (cohomological, first quadrant) *spectral sequence* consists of

- objects $E_r^{pq} \in \mathcal{A}$ for all $p, q, r \geqslant 0$
- morphisms $d_r = d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ with $d_r^2 = 0$
- isomorphisms $\ker d_r^{pq} / \operatorname{im} d_r^{p-r,q+r-1} \cong E_{r+1}^{pq}$

$$
\begin{tikzcd}
 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
E_0^{pq} & \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] \\
 & \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] \\
 & \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u]
\end{tikzcd}
$$

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$$\begin{array}{cccccccccccc}
 & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
E_1^{pq} & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
 & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
 & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet
\end{array}$$

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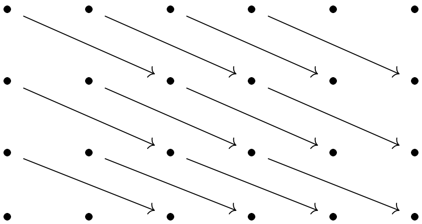
E_2^{pq}



Spectral sequences: definition

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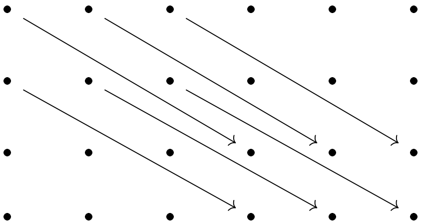
- isomorphisms $\ker d_r^{pq} / \operatorname{im} d_r^{p-r,q+r-1} \cong E_{r+1}^{pq}$

E_3^{pq}



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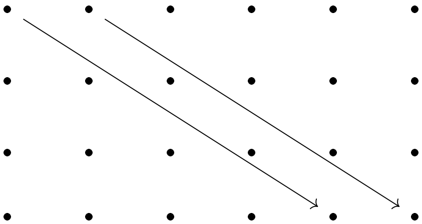
- objects $E_r^{pq} \in \mathcal{A}$ for all $p, q, r \geqslant 0$
- morphisms $d_r = d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ with $d_r^2 = 0$
- isomorphisms $\ker d_r^{pq} / \operatorname{im} d_r^{p-r,q+r-1} \cong E_{r+1}^{pq}$

E_4^{pq}



Spectral sequences: definition

Fix (p,q). Once r is sufficiently large, E_{r+1}^{pq} is computed from the complex

$$0 \xrightarrow{d_r^{p+r,q-r-1}} E_r^{pq} \xrightarrow{d_r^{pq}} 0.$$

Therefore we have canonical isomorphisms $E_r^{pq} \cong E_{r+1}^{pq} \cong E_{r+2}^{pq} \cong \dots$ We denote this limit object by E_∞^{pq}. Back to our objective of computing the cohomology of the total complex: it turns out that there is a decreasing filtration on $H^n = H^n(E^\bullet)$

$$H^n = F^0 H^n \supseteq F^1 H^n \supseteq \dots \supseteq F^n H^n \supseteq F^{n+1} H^n = 0$$

such that $E_\infty^{p,n-p} \cong F^p H^n / F^{p+1} H^n = \operatorname{gr}_p H^n$. Therefore $\bigoplus_{p=0}^n E_\infty^{p,n-p} \cong \operatorname{gr} H^n$. We say that our spectral sequence *converges* to $H^n(E^\bullet)$ and denote this $E_0^{pq} \Rightarrow H^{p+q}(E^\bullet)$. We didn't quite compute the cohomology of $E^\bullet$, but we came close. If there is an $r \geqslant 2$ such that E_r^{pq} has only one non-zero column or row, we really can read off $H^n(E^\bullet)$. We say the spectral sequence *collapses* at page r. In most applications, spectral sequences already collapse at E_1 or E_2.

In our definition of a spectral sequence, we can reverse the roles of d_h and d_v. This gives a new spectral sequence that also converges to the cohomology of the total complex, but (possibly) with a different filtration. We denote the previous definition by $\hat{E}^{pq}$ and the new definition by $\vec{E}^{pq}$. This observation is already very powerful.

EXAMPLE. the Five Lemma. Assume both rows are exact, all squares commute, and α, β, δ, and ϵ are isomorphisms. Then γ is an isomorphism.

$$\begin{array}{ccccccccc} F & \longrightarrow & G & \longrightarrow & H & \longrightarrow & I & \longrightarrow & J \\ {\scriptstyle\alpha}\uparrow & & {\scriptstyle\beta}\uparrow & & {\scriptstyle\gamma}\uparrow & & {\scriptstyle\delta}\uparrow & & {\scriptstyle\epsilon}\uparrow \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \end{array}$$

Consider the diagram as $\vec{E}_0^{pq}$. We take horizontal cohomology to get to $\vec{E}_1^{pq}$. Since the rows are exact, $\vec{E}_1^{pq}$ looks like

$$\begin{array}{ccccc} ? & 0 & 0 & 0 & ? \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ ? & 0 & 0 & 0 & ? \end{array}$$

In particular, the cohomology of the total complex vanishes in the two degrees corresponding to C and H. $\vec{E}_2^{pq}$ looks similar and the spectral sequence converges there, since there will be no more morphisms between two non-zero objects.

Now compute using the other orientation, taking the vertical differentials first. Since α, β, δ, and ϵ are all isomorphisms, $\hat{E}_1^{pq}$ looks like

$$0 \longrightarrow 0 \longrightarrow ? \longrightarrow 0 \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow ? \longrightarrow 0 \longrightarrow 0$$

and we see that the spectral sequence already converges on this page. We want to show that both question marks are zero. But they equal those cohomology objects of the total complex that are zero by the previous calculation. We're done!

Let's show that a short exact sequence of chain complex gives a long exact sequence in cohomology:

$$\begin{tikzcd}
0 \arrow[r] & A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & 0 \\
0 \arrow[r] & A_1 \arrow[r] \arrow[u] & B_1 \arrow[r] \arrow[u] & C_1 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A_0 \arrow[r] \arrow[u] & B_0 \arrow[r] \arrow[u] & C_0 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Spectral sequences: first applications

Taking horizontal cohomology, we get 0 in every position by exactness and the spectral sequence converges. Thus the cohomology of the total complex is trivial.
Taking instead vertical cohomology first, $\hat{E}_1^{pq}$ looks like

$$
\begin{array}{c}
0 \longrightarrow H^2(A) \xrightarrow{\alpha_2} H^2(B) \xrightarrow{\beta_2} H^2(C) \longrightarrow 0 \\[1em]
0 \longrightarrow H^1(A) \xrightarrow{\alpha_1} H^1(B) \xrightarrow{\beta_1} H^1(C) \longrightarrow 0 \\[1em]
0 \longrightarrow H^0(A) \xrightarrow{\alpha_0} H^0(B) \xrightarrow{\beta_0} H^0(C) \longrightarrow 0
\end{array}
$$

The next page $\hat{E}_2^{pq}$ looks like

$$
\begin{tikzcd}
0 & \ker\alpha_2 \arrow[ddrr] & \ker\beta_2/\operatorname{im}\alpha_2 & \operatorname{coker}\beta_2 & 0 \\
0 & \ker\alpha_1 \arrow[ddrr] & \ker\beta_1/\operatorname{im}\alpha_1 & \operatorname{coker}\beta_1 & 0 \\
0 & \ker\alpha_0 & \ker\beta_0/\operatorname{im}\alpha_0 & \operatorname{coker}\beta_0 & 0
\end{tikzcd}
$$

The diagonal arrows drawn are the only non-zero morphisms on this page or on any subsequent page. So the spectral sequence will converge on the next page. We know it must converge to zero in every entry. This can only happen if $\ker\beta_j/\operatorname{im}\alpha_j = 0$ and if the diagonal arrows are isomorphisms. This gives the desired long exact sequence.

Spectral sequences: first applications

One more example: the Snake lemma. Consider the following diagram, where rows are exact and squares commute.

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r] & E \arrow[r] & F \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

We want to prove the exactness of

$$0 \longrightarrow \ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \longrightarrow \operatorname{im}\alpha \longrightarrow \operatorname{im}\beta \longrightarrow \operatorname{im}\gamma \longrightarrow 0.$$

Taking horizontal cohomology, we see that all entries of $\vec{E}_1^{pq}$ are zero and the spectral sequence has converged. On the other hand, $\hat{E}_1^{pq}$ looks like

$$0 \longrightarrow \operatorname{im}\alpha \longrightarrow \operatorname{im}\beta \longrightarrow \operatorname{im}\gamma \longrightarrow 0$$

$$0 \longrightarrow \ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \longrightarrow 0.$$

Taking horizontal cohomology, we get $\hat{E}_2^{pq}$:

$$\begin{tikzcd}
0 \arrow[drr] & 0 \arrow[drr] & & & & & \\
& 0 \arrow[drr] & ?? \arrow[drr] & ? \arrow[drr] & ? & 0 & \\
& 0 & ? & ? \arrow[drr] & ?? \arrow[drr] & 0 & \\
& & & & & 0 & 0
\end{tikzcd}$$

The single question marks converge on this page. Therefore they must equal zero. The double question marks will converge on the next page, as there will be no more non-zero arrows into or out of them. They also must be zero on $\hat{E}_3$, so the arrow between them is an isomorphism, i.e. $\operatorname{coker}(\ker\beta \to \ker\gamma) \cong \ker(\operatorname{im}\alpha \to \operatorname{im}\beta)$. This proves the Snake lemma.

Spectral sequences: properties

From now on, we consider only spectral sequences of type $\hat{E}_r^{pq}$, i.e. with d_0 vertical. We also abstract slightly: we will let a spectral sequence converge to any family of finitely filtered objects $E^n \in \mathcal{A}$ with $F^0E^n = E^n$ and $F^{n+1}E^n = 0$, i.e. $E_\infty^{pq} \cong \operatorname{gr}_p E^{p+q}$.
Notice that each E_{r+1}^{pq} is a subquotient of E_r^{pq}. In particular there are natural quotient maps

$$E_0^{n,0} \to E_1^{n,0} \to \dots \to E_\infty^{n,0}.$$

Now $E_\infty^{n,0} \xrightarrow{\cong} \operatorname{gr}_n E^n = F^nE^n/F^{n+1}E^n = F^nE^n \hookrightarrow E^n$.
The composite $E_0^{n,0} \to E^n$ is an *edge morphism*.
E^n naturally surjects onto $E_\infty^{0,n} = F^0E^n/F^1E^n = E^n/F^1E^n$, and $E_\infty^{0,n}$ naturally injects into $E_r^{0,n}$, which injects into $E_{r-1}^{0,n}$. The composite $E^n \to E_0^{0,n}$ is the other edge morphism.
EXERCISE. (5 term exact sequence) the following sequence is exact:

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \xrightarrow{d} E_2^{2,0} \to E^2$$

EXAMPLE. The Hochschild-Serre spectral sequence computes the group cohomology of G in terms of the cohomology of G/H and H: $H^p(G/H, H^q(H,A)) \Rightarrow H^{p+q}(G,A)$. In this case, the 5 term exact sequence is the inflation-restriction exact sequence

$$0 \to H^1(G/H, A^H) \to H^1(G,A) \to H^1(H,A)^{G/H} \to H^2(G/H, A^H) \to H^2(G,A).$$

THEOREM (GROTHENDIECK). Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be abelian categories such that both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Let there be left exact functors $G : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{C}$. Call an object B of $\mathcal{B}$ *F-acyclic* if the derived functors of F vanish on B, i.e. $R^i F(B) = 0$ for $i \neq 0$. Suppose that G sends injective objects of $\mathcal{A}$ to F-acyclic objects of B. Then there exists a convergent (first quadrant, cohomological) spectral sequence, beginning on page 2, for each $A \in \mathcal{A}$:

$$E_2^{pq} = (R^p)(R^q)(A) \Rightarrow R^{p+q}(FG)(A)$$

PROOF (sketch). For $A \in \mathcal{A}$, take in injective resolution $A \to I^\bullet$ and apply G to it to get a cochain complex in $\mathcal{B}$. Take an (appropriate) injective resolution to get a first quadrant double complex. Apply F to the double complex. The cohomology of the total complex of the resulting double complex is denoted $\mathbb{R}^n F(G(I^\bullet))$, the right hyper-derived functors of F. As before, we have two different spectral sequences both converging to $(\mathbb{R}^n F)(G(I^\bullet))$.

The Grothendieck spectral sequence

Starting with the injective resolution of $G(I^\bullet)$, we can take vertical cohomology first and horizontal cohomology second to get

$$\hat{E}_2^{pq} = H^p((R^q F)(G(I^\bullet))) \Rightarrow (\mathbb{R}^{p+q} F)(G(I^\bullet)).$$

Reversing that order, we get

$$\vec{E}_2^{pq} = (R^p F) H^q(G(I^\bullet)) \Rightarrow (\mathbb{R}^{p+q} F)(G(I^\bullet)).$$

By hypothesis, each $G(I^p)$ is F-acyclic, so $(R^q F)(G(I^p)) = 0$ for $q \neq 0$. Thus $\hat{E}_2^{pq}$ consists of a single row and the spectral sequence collapses, giving

$$\mathbb{R}^p F(G(I^\bullet)) \cong H^p((R^0 F) G(I^\bullet)) = H^p(FG(I^\bullet)) = R^p(FG)(A).$$

Now the other spectral sequence looks like

$$\vec{E}_2^{pq} = (R^p F) H^q(G(I^\bullet)) \Rightarrow R^p(FG)(A).$$

Since $H^q(G(I^\bullet)) = R^q G(A)$, we are done.

References

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