Étale cohomology reading seminar

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1 Introduction

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1.1 Why study étale cohomology?

Fix a field k, and let $f \in k[x_0, \ldots, x_{n+1}]$ be a homogeneous irreducible polynomial with $(\frac{\mathrm{d}f}{\mathrm{d}x_0}, \ldots \frac{\mathrm{d}f}{\mathrm{d}x_{n+1}}) \neq (0, \ldots, 0)$. Then the zero-set of f over k, $X(k)$, determines a subset of $\mathbb{P}^{n+1}_k$, and very loosely speaking, the goal of algebraic geometry is to understand $X(k)$.

One powerful method of studying $X(k)$ is through its invariants. For example, if $k = \mathbb{C}$, then $X(\mathbb{C}) \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is naturally a complex manifold of real dimension $2n$, and we can define the *singular cohomology groups* $H^i(X(\mathbb{C}); \mathbb{Z})$ for $i = 0, \ldots, 2n$. Then

$$H^i(X(\mathbb{C}); \mathbb{Q}) := H^i(X(\mathbb{C}); \mathbb{Z}) \otimes \mathbb{Q}$$

form $\mathbb{Q}$-vector spaces, and $b_i := \dim_{\mathbb{Q}} H^i(X(\mathbb{C}); \mathbb{Q})$ is called the *i-th Betti number* of $X(\mathbb{C})$. The *Euler characteristic* of $X(\mathbb{C})$ is defined to be $\chi(X) := \sum_{i=0}^{2n} (-1)^i b_i$.

Example 1.1. If $\deg f = 1$, then $X(\mathbb{C}) \cong \mathbb{P}^n_{\mathbb{C}}$, and one can compute that

$$b_i = \begin{cases} 1 & i \leq 2n \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

so the Euler characteristic of $X(\mathbb{C})$ is $n + 1$.

Example 1.2. If $n = 1$ and $d := \deg f \geq 2$, then the Riemann surface $X(\mathbb{C})$ has genus $g = \frac{(d-1)(d-2)}{2}$, meaning $X(\mathbb{C})$ is homeomorphic to a donut with g holes or a sphere with g handles. One can show that $(b_0, b_1, b_2) = (1, 2g, 1)$, so $\chi(X) = 2-2g$.

For example, if $d = 3$ then X is an elliptic curve, with genus $g = 1$ and Euler characteristic $\chi(X) = 0$.

These examples show that we can use topology to distinguish between different $X(k)$ when $k = \mathbb{C}$. However, if k is a finite field there are no such topological invariants. More precisely, if $k = \mathbb{F}_q$ where q is a prime power, then $X(k)$ is a finite set, and naively the only reasonable invariant we can define is the number of points. Let $N_r(X) := \#X(\mathbb{F}_{q^r})$ be the number of points of X defined over $\mathbb{F}_{q^r}$.

Example 1.3. If $X = \mathbb{P}^n_{\mathbb{F}_q}$, then it is straightforward (exercise!) to show that

$$N_r(\mathbb{P}^n_{\mathbb{F}_q}) = \frac{(q^r)^{n+1} - 1}{q^r - 1} = (q^r)^n + (q^r)^{n-1} \dots + q^r + 1 \tag{1.1}$$

Example 1.4. Suppose X is an elliptic curve over $\mathbb{F}_{q^r}$. Then *Hasse's theorem* gives a good estimate of each N_r:

$$|N_r(X) - q^r - 1| \leq 2q^{r/2}. \tag{1.2}$$

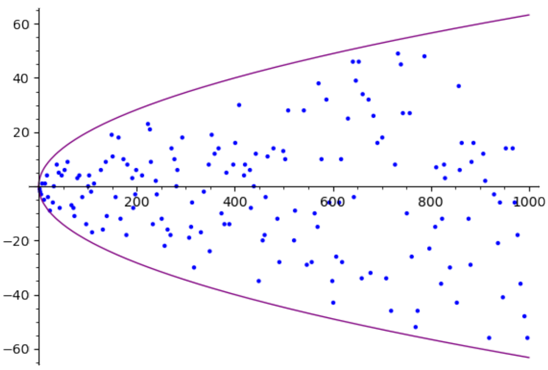


Figure 1: The value of $\#X(\mathbb{F}_p) - p - 1$ as p ranges between 1 and 1000, where X is the elliptic curve $y^2 = x^3 - 2619x + 54486$, with Hasse's bound $\pm 2\sqrt{p}$ in purple.

Weil found the following generalisation of Hasse's result:

Theorem 1.5 (Weil)**.** *Let X be a non-singular projective curve of genus g defined over $\mathbb{F}_q$. Then there exist algebraic integers $a_1, \dots, a_{2g}$ such that:*

(i) *For every $r \geq 1$,*
$$N_r(X) = q^{r/2} + 2 - (a_1^r + \dots + a_{2g}^r), \tag{1.3}$$

(ii) *the numbers $\{a_i\}$ are q-*Weil numbers of weight 1*, that is, $|a_i| = q^{1/2}$ for $1 \leq i \leq 2g$.*

One easily checks that this implies the Hasse-Weil theorem. Note that the property of being a q-Weil number is quite restrictive; $a_i = 3 \pm 2i\sqrt{2}$ is an example of such a number, when $q = 17$.

Taking $k = \mathbb{Q}$, let's assume furthermore that $f \in \mathbb{Z}[x_0, \dots, x_{n+1}]$ and is primitive, and suppose the reduction mod q, $\overline{f}$ defines an irreducible and smooth $X(\mathbb{F}_q)$. For convenience, we will denote such a model of X by $\mathcal{X}$. Then we have an informal diagram

$$\begin{tikzcd} & \mathcal{X}/\mathbb{Z} \arrow[dl] \arrow[dr] & \\ \mathcal{X}(\mathbb{F}_q)/\mathbb{F}_q \arrow[rr, dashed, leftrightarrow, "?"] & & \mathcal{X}(\mathbb{C})/\mathbb{C} \end{tikzcd} \tag{1.4}$$

It is natural to ask whether there is any interaction between the structures of X over $\mathbb{F}_q$ and $\mathbb{C}$, as indicated by the arrow marked "?". For example, we might hope that there is a connection between the invariants N_r and b_i or χ. Note that the structure on the left is fundamentally arithmetic, being defined mod q, whereas the right hand side is topological.

1.2 The Weil conjectures

A satisfactory answer to this question was conjectured by Weil, and is one of the most stunning applications of étale cohomology. First we need to describe the setup:

Definition 1.6. Let $X/\mathbb{F}_q$ be as in the previous section. The *zeta function of X* is the formal power series $\zeta(X,T) \in \mathbb{Q}[[T]]$ defined by

$$\zeta(X,T) = \exp\left(\sum_{r\geq 1} \frac{N_r(X)}{r} T^r\right), \tag{1.5}$$

where $\exp(x) = \sum_{n\geq 0} x^n/n!$ is the formal exponential series.

This might look like an arbitrary definition at first, but note that $\frac{d}{dT}\log\zeta(X,T) = \sum N_{r+1}T^r$, which is the generating function of $N_r(X)$.

Example 1.7. When $X = \mathbb{P}^n_{\mathbb{F}_q}$, it is a fun exercise to show that

$$\zeta(\mathbb{P}^n,T) = \frac{1}{(1-T)(1-qT)\dots(1-q^nT)}, \tag{1.6}$$

(Hint: use eq. (1.1) and expand the resulting exponentials.)

Example 1.8. With a similar argument using Weil's theorem 1.5, one can check that if X is a curve of genus g, then

$$\zeta(X,T) = \frac{(1-a_1T)\dots(1-a_{2g}T)}{(1-T)(1-qT)}. \tag{1.7}$$

In both of the examples, we see that while $\zeta(X,T)$ is originally defined as a power series, it is actually a rational function in T defined over some finite extension of $\mathbb{Q}$. We also see that the degree of the numerator is the sum of the odd Betti numbers, while the denominator has degree the sum of the even.

Theorem 1.9 (Weil conjectures). *Let $X/\mathbb{F}_q$ be a smooth projective variety of dimension n.*

(I) $\zeta(X,T)$ is a rational function; in fact

$$\zeta(X,T) = \frac{Q_1Q_3\cdots Q_{2n-1}}{Q_0Q_2\cdots Q_{2n}}, \tag{1.8}$$

where $Q_i \in \mathbb{Z}[T]$ are given by $Q_i := \prod_{j=1}^{b_i}(1-a_{ij}T)$ for some $b_i \in \mathbb{N}$ and algebraic integers a_{ij}.

(II) $\zeta(X,T)$ satisfies the functional equation $\zeta(X,1/q^nT) = \pm q^{\chi n/2}T^{\chi}\zeta(X,T)$ and $\chi := \sum(-1)^i b_i$.

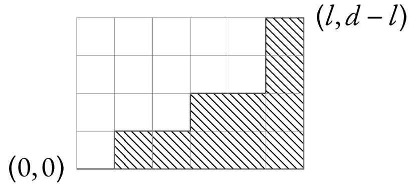
(III) The numbers a_{ij} are q-Weil numbers of weight i, meaning $|a_{ij}| = q^{i/2}$ for all $1 \le i,j \le 2n$.

(IV) If X has a "nice" model $\mathcal{X}$ defined over $\mathbb{Z}$, then b_i are precisely the Betti numbers of $\mathcal{X}(\mathbb{C})$.

(III) is referred to as the "Riemann hypothesis" for $\zeta(X,T)$, since it tells precisely where its zeroes and poles lie. Historically, the first progress on the Weil conjectures was made by Dwork who proved (I) using p-adic analytic methods. Another proof of this came with the full proof of the Weil conjectures through the work of Grothendieck and Artin, and Deligne.

While it is astonishing that information about the topology of $\mathcal{X}(\mathbb{C})$ determines the number of points when reducing modulo a prime, it is also possible to obtain information the other way, through so-called "point counting". In a sense, this is a local-to-global principle; the "global" information about the topology of $\mathcal{X}(\mathbb{C})$ is determined by "local data".

Exercise. Let $G(l,d)$ denote the complex Grassmannian (definition). Using theorem 1.9, show that $b_i(G(l,d))$ equals the number of paths on a grid from $(0,0)$ to $(l,d-l)$ with area i (where we can only move right or up).



Of course, these Betti numbers were known before the Weil conjectures were established. However, for more complicated varieties the point-counting method is sometimes one of the easiest ways of determining the topological structure.

1.3 You could have invented étale cohomology[1]

In the previous section we saw that several topological invariants of varieties over $\mathbb{C}$ defined in terms of cohomology were related to invariants over $\mathbb{F}_q$. Weil suggested that this could be possible using a "good" cohomology theory for varieties over $\mathbb{F}_q$.

Fix a topological space T and an abelian group A. We want to find a definition of cohomology groups $H^\bullet(T, A)$ which gives a meaningful answer in the context of algebraic geometry.

Example 1.10 (Singular cohomology)**.** Choosing $H^\bullet(T, A)$ to be singular cohomology doesn't work in general, because there are too few continuous maps in the Zariski topology.

Example 1.11 (Sheaf cohomology)**.** Let X be an irreducible scheme and $\underline{A}$ the locally constant sheaf $X \supset U \mapsto A$. Then for any pair of open sets $U, V \subset X$ with $U \subset V$ we have $\underline{A}(V) \cong A \xrightarrow{\sim} \underline{A}(U)$, and so in particular $\underline{A}$ is *flabby*, which implies that the sheaf cohomology groups $H^i(X, \underline{A})$ vanish for $i > 0$ (see [Har77, Ex. III 2.3]).

To remedy this, we first recall some sheaf theory: given a topological space T, let Op/T be the category where the objects are open subsets of T, and morphisms are given by inclusions $U \hookrightarrow V$ whenever $U \subset V$ for $U, V \in \mathrm{Op}/T$. A *presheaf* is a contravariant functor $\mathcal{F}\colon \mathrm{Op}/T \to \mathrm{Ab}$, where Ab denotes the category of abelian groups. A presheaf $\mathcal{F}$ is a *sheaf* if it satisfies the *sheaf condition*: for any $U \in \mathrm{Op}/T$ and any covering $\bigcup_i U_i$ of U with $U_i \in \mathrm{Op}/T$, we have an *equaliser diagram*:

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j). \tag{1.9}$$

Since $U_i \hookrightarrow U$, we have maps $\rho_i\colon \mathcal{F}(U) \to \mathcal{F}(U_i)$ which assemble to the first map of the diagram: $u \mapsto (\rho_i(u))_i$. The double arrows are $(u_i)_i \mapsto (\rho_{i,j}(u_i))$ and

[1] The title is a reference to Timothy Chow's paper "You could have invented spectral sequences", see here [Cho06].

$(u_i)_i \mapsto (\rho_{i,j}(u_j))$, respectively, where $\rho_{i,j}\colon U_i \to U_i \cap U_j$. Equation (1.9) being an equaliser diagram in this case simply means that $\mathscr{F}(U)$ is the kernel of the difference of the two maps on the right.

The crucial idea is that to define sheaves, we actually don't need the full power of a topology, but rather just the notion of coverings. Regarding $U_i \cap U_j$ as the categorical fibre product $U_i \times_T U_j$, we can replace Op/T with the category of whose objects are topological spaces U equipped with a local homeomorphism $U \to T$, and where morphisms are continuous maps which factor through these. Let us denote this by $\text{Ét}/T$. As before, a sheaf is any presheaf that satisfies the equaliser condition, eq. (1.9).

Proposition 1.12. *There is an equivalence of categories* $\mathrm{Sh}(\mathrm{Op}/T) \xrightarrow{\sim} \mathrm{Sh}(\text{Ét}/T)$.

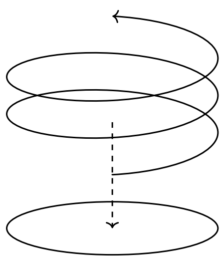
Proof. □

We can summarise this by saying that the main conceptual leap required to define étale cohomology was to replace open subsets and inclusions by coverings and local homeomorphisms.

Of course, it remains to find a good notion of a "local homeomorphism" between schemes.

Attempt 1:

Let $f\colon X \to Y$ be a morphism of schemes, and suppose we mimic the definition of a local homeomorphism and require that for any $x \in X$ we can find a Zariski-open neighbourhood U such that $f|_U\colon U \to f(U)$ is an isomorphism onto an open subscheme of Y. This is too rigid; by analogy with covering spaces of Riemann surfaces we would like the map $\mathbb{G}_m \to \mathbb{G}_m$ defined by $t \mapsto t^n$ (this corresponds to the map $z \mapsto z^n$ from $\mathbb{C}^\times$ to itself) to be a "covering map" in a suitable sense.



However, an open set in $\mathbb{G}_m$ is given by the complement of finitely many closed points, and such an open set upstairs does not look like an open downstairs.

This example also rules out the requirement that f should be an isomorphism at the stalks. Indeed, looking at the stalk at 1, we easily see that the induced map

$k[t,t^{-1}]_{(t-1)} \to k[t,t^{-1}]_{(t-1)}$ given by $t \mapsto t^n$ is not surjective. The slogan here is that "stalks know too much about the global structure". However, since we are looking for a local condition, it makes sense to look for something defined in terms of stalks.

Attempt 2:

Suppose that for all closed points $x \in X$, we require the induced maps on completions $\widehat{\mathcal{O}}_{Y,f(x)} \to \widehat{\mathcal{O}}_{X,x}$ to be isomorphisms. One motivation behind this that the completion $\widehat{\mathcal{O}}_{X,x}$ "knows less than $\mathcal{O}_{X,x}$" in some sense.

Exercise. Let k be an algebraically closed field, and let $f : X \to Y$ be a morphism of smooth k-varieties (smooth separated integral finite-type k-schemes). Then the following are equivalent:

(i) For all closed points x in X, the map of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induces an isomorphism $\widehat{\mathcal{O}}_{Y,f(x)} \to \widehat{\mathcal{O}}_{X,x}$ on the completions.
(ii) For all closed points x in X, the morphism $Tf : T_xX \to T_{f(x)}Y$ on tangent spaces is an isomorphism.
(iii) If $k = \mathbb{C}$, $f : X(\mathbb{C}) \to Y(\mathbb{C})$ is a local isomorphism of smooth manifolds.

This gives rise to the notion of a morphism being *formally étale*, first coined by Grothendieck. The roadmap for developing étale cohomology is now:

(i) Develop a good theory of étale morphisms.
(ii) Develop sheaf theory in terms of covers, not opens.
(iii) Apply the two former to compute things.

Time permitting, we can look at other interesting applications.

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