FUNDAMENTAL CLASS & THE CYCLE MAP

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Recall: On Tuesday Andrés proved a partial result of the purity thm.:

$\operatorname{Spec} k$, $k = k^{sep}$, $\operatorname{char} k \nmid n$

> Let (Z, X) be a smooth $\overset{\shortparallel}{S}$-pair of codimension c, and let $\mathcal{F} \in \underline{Sh}(X_{et})$ be locally isomorphic to Λ ($\Lambda = \mathbb{Z}/n\mathbb{Z}$, $\operatorname{char} X \nmid n$). Then
>
> $\Lambda = \mathbb{Z}/n\mathbb{Z}$
>
> $$\underline{H}^r_Z(X, \mathcal{F}) := R^r i^! \mathcal{F} = \begin{cases} 0 & \text{if } r \neq 2c, \\ \text{locally isomorphic to } \Lambda & \text{if } r = 2c. \end{cases}$$

Goal: improve this to say $\underline{H}^{2c}_Z(X, \Lambda(c)) \cong \Lambda$

To do that, it is enough to find an element of order n in

$$\Gamma(Z, \underline{H}^{2c}_Z(X, \Lambda(c))) = H^{2c}_Z(X, \Lambda(c)) \ni s_{Z/X}$$

FUNDAMENTAL CLASS OF Z IN X

already proved by Andrés

Indeed, such an element will define a map $\Lambda \longrightarrow \underline{H}^{2c}_Z(X, \Lambda(c))$, which will be an isomorphism on stalks.

We begin by constructing $s_{Z/X}$ in the case $c=1$ and Z irreducible, so that Z is a smooth prime divisor of X. ($U = X \setminus Z$)

$$\begin{tikzcd}
H^0(X, \mathbb{G}_m) \arrow[r, "a"] \arrow[d, leftrightarrow, "\sim"] & H^0(U, \mathbb{G}_m) \arrow[r, "b"] \arrow[d, leftrightarrow, "\sim"] \arrow[dr, phantom, "\circlearrowright"] & H^1_Z(X, \mathbb{G}_m) \arrow[r, "c"] \arrow[dr, phantom, "\circlearrowright"] & H^1(X, \mathbb{G}_m) \arrow[r, "d"] \arrow[d, leftrightarrow, "\sim"] & H^1(U, \mathbb{G}_m) \arrow[d, leftrightarrow, "\sim"] \\
\Gamma(X, \mathcal{O}_X^*) \arrow[r, "a'"] & \Gamma(U, \mathcal{O}_U^*) \arrow[r, "b'", "\operatorname{ord}_Z"'] & \mathbb{Z} \arrow[u, "\sim"] \arrow[r, "c'"] & \operatorname{Pic}(X) \arrow[r, "d'"] & \operatorname{Pic}(U)
\end{tikzcd}$$

$1 \longmapsto [Z]$

$$0 \longrightarrow \operatorname{im} a = \ker b \longrightarrow H^1_Z(X, \mathbb{G}_m) \longrightarrow \operatorname{im} c = \ker d \longrightarrow 0$$

$\operatorname{im} a' = \ker b' = \mathbb{Z}$ or 0 $\qquad$ $\mathbb{Z}/n\mathbb{Z}$ or $0 = \operatorname{im} c' = \ker d'$

If either of them is zero, we may directly conclude $H^1_Z(X,\mathbb{G}_m)=\mathbb{Z}$.
Otherwise we see a s.e.s $[0\to\mathbb{Z}\to H^1_Z(X,\mathbb{G}_m)\to \mathbb{Z}/n\mathbb{Z}\to 0.]$

Recall: $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$

Consider the LES in H_Z^* associated to the Kummer sequence

$$
\begin{tikzcd}
0 \arrow[r] & H^0_Z(X,\mu_n) \arrow[r] & H^0_Z(X,\mathbb{G}_m) \arrow[r] & H^0_Z(X,\mathbb{G}_m) \arrow[dll, bend left] \\
 & \cancel{H^1_Z(X,\mu_n)} \arrow[r] & H^1_Z(X,\mathbb{G}_m) \arrow[r, "n", "\text{injective}"'] & H^1_Z(X,\mathbb{G}_m)
\end{tikzcd}
$$

(purity) 0 $\qquad H^1_Z(X,\mu_n) = \Lambda(1)$

So we have a morphism of short exact sequences + snake lemma

$$
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[ddll, bend left] \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "n"] & H^1_Z(X,\mathbb{G}_m) \arrow[r] \arrow[d, "n"] & \mathbb{Z}/n\mathbb{Z} \arrow[r] \arrow[d, "n"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & H^1_Z(X,\mathbb{G}_m) \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0 \\
 & \mathbb{Z}/n\mathbb{Z} \arrow[r] & \mathrm{Coker}\, n \arrow[r, "\sim"] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0
\end{tikzcd}
$$

Since $\mathbb{Z}/n\mathbb{Z}$ is finite, injective $\Rightarrow$ bijective and we conclude that the cokernel of multiplication by n on $H^1_Z(X,\mathbb{G}_m)$ is $\mathbb{Z}/n\mathbb{Z}$

$$0\to \mathbb{Z}\xrightarrow{\text{in}} H^1_Z(X,\mathbb{G}_m)\xrightarrow{\text{ke}} \mathbb{Z}/n\mathbb{Z}\to 0 \quad\Longrightarrow\quad H^1_Z(X,\mathbb{G}_m) = \mathbb{Z}\oplus T,\ T \text{ torsion}$$

Since the cokernel of multiplication by n is $\mathbb{Z}/n\mathbb{Z}$, we must have $T/nT=0$.
But $nT=0$ because $\forall t\in T\ [nt]=0\in \mathbb{Z}/n\mathbb{Z} \Rightarrow nt$ comes from $\mathbb{Z}$ by exactness, but injectivity + torsion $\Rightarrow nt=0$ $\quad$ Hence $T=0$

$nT=0$

$$H^1_Z(X,\mathbb{G}_m)\cong\mathbb{Z}$$

So now we may consider once again the Kummer LES for H_Z^*

$$\begin{array}{ccccc} \mathbb{Z} & & \mathbb{Z} & & \mu_n \\ \wr\!\shortparallel & & \wr\!\shortparallel & & \shortparallel \\ H_Z^1(X,\mathbb{G}_m) & \xrightarrow{n} & H_Z^1(X,\mathbb{G}_m) & \xrightarrow{\delta} & H_Z^2(X,\Lambda(1)) \\ & & 1 & \xmapsto{def} & s_{Z/X} \end{array}$$

THEOREM.– [EC, VI.6.1] There is a unique function $(Z,X) \mapsto s_{Z/X}$ associating to each smooth S-pair of codimension c a FUNDAMENTAL CLASS $s_{Z/X} \in H_Z^{2c}(X,\Lambda(c))$ satisfying the following:

(a) $s_{Z/X}$ has order n

(b) If $c=1$ and Z is irreducible, $s_{Z/X}$ is as defined above

(c) If $\phi:(Z',X') \to (Z,X)$ is a morphism of smooth S-pairs of codimension c ($Z' = Z \times_X X'$) then $\phi^*(s_{Z/X}) = s_{Z'/X'}$

$\phi^* : H_Z^{2c}(X; \Lambda(c)) \longrightarrow H_{Z'}^{2c}(X', \Lambda(c))$ $\quad Z = \coprod Z_j \implies H_Z^{2c}(X) = \bigoplus H_{Z_j}^{2c}(X)$

(d) If
$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "i"'] & Y \arrow[dl, "u"] \\ X & \end{tikzcd}$$
are smooth pairs of codimensions $\underset{Z\subset Y}{a}, \underset{Y\subset X}{b}, \underset{Z\subset X}{c}$ then there is an isomorphism

$H_Z^{2c}(X,\Lambda(c)) \cong H_Z^{2a}(Y,\Lambda(a)) \otimes H_Y^{2b}(X,\Lambda(b))$

and under this isomorphism $s_{Z/X} = s_{Z/Y} \otimes s_{Y/X}$

And with this, we have proved our "baby" Purity Theorem.

It is worth noting that according to [EC, VI.6.5.(a)], we may express the fundamental class using the Gysin map

$$\left[\begin{array}{ccc} H^0(Z,\Lambda) \cong \Lambda & \xrightarrow{\text{Gysin}} & H^{2c}(X,\Lambda(c)) \\ 1 & \xmapsto{\text{thm}} & s_{Z/X} \end{array}\right]$$

Now we are going to use the fundamental class with a different purpose

Our goal is to define a functorial graded ring homomorphism

$$cl_X : CH^*(X) \longrightarrow H^{2*}(X), \qquad H^*(X) := \bigoplus H^r(X, \Lambda(\lfloor \tfrac{r}{2} \rfloor))$$

CYCLE MAP

So we are going to review the Chow group and the ring structure on H^*.

CUP-PRODUCT: Using [EC, V.1.16] we may extend

$$\left[\begin{array}{ccc} \Gamma(X,\mathcal{F}_1)\times\Gamma(X,\mathcal{F}_2) & \longrightarrow & \Gamma(X,\mathcal{F}_1\otimes\mathcal{F}_2) \\ (s_1, s_2) & \longmapsto & s_1\otimes s_2 \end{array}\right]$$

to a functorial pairing called CUP PRODUCT (works for $R\Gamma_{\pi_*}$ and H_Z)

$$\left[\begin{array}{ccc} H^r(X,\mathcal{F}_1)\times H^s(X,\mathcal{F}_2) & \longrightarrow & H^{r+s}(X,\mathcal{F}_1\otimes\mathcal{F}_2) \\ (\gamma_1, \gamma_2) & \longmapsto & \gamma_1\cup\gamma_2 \end{array}\right]$$

which satisfies $\rightsquigarrow d\gamma_1 \cup \gamma_2 = d(\gamma_1\cup\gamma_2)$ $\qquad \gamma_1\cup\gamma_2 = (-1)^{rs}\gamma_2\cup\gamma_1$

$\rightsquigarrow \gamma_1\cup d\gamma_2 = (-1)^r d(\gamma_1\cup\gamma_2)$

It is also possible to give other constructions that lead to this same product: Čech cohomology, Godement resolutions, Ext pairing (see [EC, 171-174]).
This makes $H^*(X)$ into an associative anticommutative graded ring.

CHOW RING: We work with quasi-projective smooth varieties over $k=k^{alg}$.

$\rightsquigarrow$ PRIME r-CYCLE := closed irreducible subvariety of codim r

$\rightsquigarrow$ r-CYCLE $\in C^r(X)$:= free ab. group generated by prime r-cycles

$\rightsquigarrow$ CYCLE $\in C^*(X) := \bigoplus_r C^r(X)$

For any closed subscheme Z of codimension r, let $Y_1,\dots,Y_s$ be the irreducible components of codim r and define the cycle associated to Z

$$Z = \sum_{i=1}^{s} n_i Y_i, \qquad n_i = \text{length } \mathcal{O}_{Y_i,Z}$$

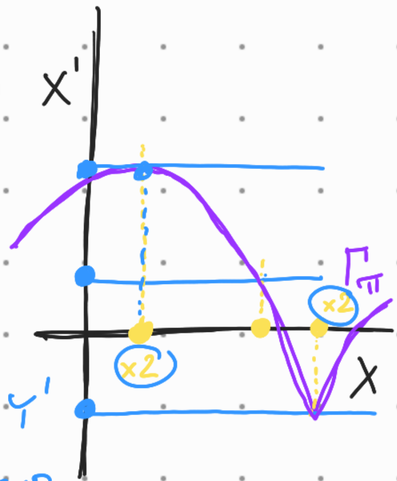
For any morphism of varieties $\pi : X \to X'$ we define two operations

$Y \subset X$ prime r-cycle $\rightsquigarrow$ $\pi_* Y = \begin{cases} 0 & \text{if } \dim \pi(Y) < \dim Y \\ \underbrace{[K(Y) : K(\pi(Y))]}_{\text{degree of } \pi \text{ along } Y}\, \pi(Y) & \text{if } \dim \pi(Y) = \dim Y \end{cases}$

(extend linearly to cycles)

$Y' \subset X'$ subvariety $\rightsquigarrow$ $\pi^* Y' := p_{1,*}\left(\Gamma_\pi \cdot (X \times Y')\right)$

$$\begin{tikzcd} & X \times X' \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X' \end{tikzcd}$$



We also define the concept of rational equivalence:

$$f : \tilde{V} \xrightarrow{\text{normaliz.}} V \xhookrightarrow{\text{subvariety}} X$$

Weil = Cartier and we have linear equivalence

D, D' linearly equiv $\xRightarrow{\text{DEF}}$ $f_* D, f_* D'$ rationally equivalent

CHOW GROUP

$$CH^*(X) := C^*(X) \big/ \sim_{\text{rational equivalence}}$$

An intersection theory (on $\mathcal{B} = \left\{ \begin{smallmatrix} \text{quasi-projective} \\ \text{smooth varieties} \\ \text{over } k = k^{alg} \end{smallmatrix} \right\}$) consists of giving a pairing $CH^r(X) \times CH^s(X) \to CH^{r+s}(X)$ for each r, s and for each $X \in \mathcal{B}$ satisfying the following axioms [Har, A.1.1]

(A1) The intersection pairing makes $CH(X)$ into a commutative, associative ring with 1 for every $X \in \mathcal{B}$. This is called the CHOW RING of X.

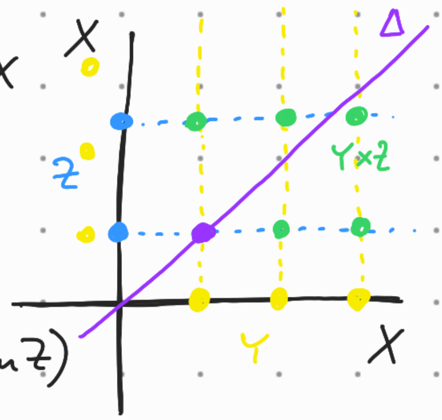
(A2) $\forall \pi : X \to X'$, $\pi^* : CH^*(X') \to CH^*(X)$ is a ring homomorphism and $\pi^* \circ \tau^* = (\tau \circ \pi)^*$

(A3) $\forall \pi : X \to X'$ proper $\pi_* : CH^*(X) \to CH^*(X')$ is a homomorphism of graded groups and $\tau_* \circ \pi_* = (\tau \circ \pi)_*$

(A4) PROJECTION FORMULA: $\forall \pi : X \to X'$ proper, $x \in CH^*(X)$, $y \in CH^*(X')$

$$\pi_*(x \cdot \pi^* y) = \pi_* x \cdot y$$

(A5) REDUCTION TO THE DIAGONAL: if Y, Z are cycles on X and $\Delta : X \to X \times X$ is the diagonal morphism then $Y \cdot Z = \Delta^*(Y \times Z)$



(A6) LOCAL NATURE: if Y, Z are subvarieties of X which intersect properly (pure $\operatorname{codim}(Y \cap Z) = \operatorname{codim} Y + \operatorname{codim} Z$)

then $Y \cdot Z = \sum \underbrace{i(Y, Z; W_j)}_{\text{local intersection multiplicity of } Z \text{ and } Y \text{ along } W_j} W_j$

where W_j are the irreducible components of $Y \cap Z$ and $i(Y, Z; W_j)$ depends only on a nbhd of the generic point of W_j on X.

(A7) NORMALIZATION: If Y is a subvariety of X and Z is an effective Cartier divisor meeting Y properly, then $Y \cdot Z$ is just the cycle associated to the Cartier divisor $Y \cap Z$ on Y.

THEOREM.- [Hart, A.1.1] There is a unique intersection theory on $\mathcal{B}$.

So now we know what it means for

$$cl_X : CH^*(X) \longrightarrow H^{2*}(X)$$

to be a functorial graded ring homomorphism.

We define the map as follows:

- if Z is nonsingular then we may associate it its fundamental class $s_{Z/X} \in H^{2c}(X, \Lambda(c)) \subset H^{2*}(X)$, which is the image of 1 under the Gysin map

$$\Lambda \cong H^0(Z, \Lambda) \longrightarrow H^{2c}(X, \Lambda(c))$$

• If Z is singular, let $Y = \operatorname{Sing} X$ be its singular locus, which is a subvariety of strictly greater codimension. Then

$k = k^{alg}$

$$\Lambda \cong H^0(Z\setminus Y, \Lambda) \overset{\text{Gysin}}{\cong} H^{2c}_{Z\setminus Y}(X\setminus Y, \Lambda(c)) \overset{(*)}{\cong} H^{2c}_Z(X, \Lambda(c)) \longrightarrow H^{2c}(X, \Lambda(c))$$

and we define $cl_X(Z)$ to be the image of 1 under these maps.

(*) LEMMA.- (Semipurity) For any closed subvariety $Z \hookrightarrow X$ of $\operatorname{codim} = c$ we have $H^r_Z(X, \Lambda) = 0 \ \forall r < 2c$.

Proof.- Z regular $\Rightarrow$ follows from Gysin. / Induction on $\dim Z$

$\dim Z = 0 \Rightarrow Z$ regular ✓

$\dim Z > 0 \Rightarrow$ take the LES associated to a triple $X \supset U \supset V$, $U = X\setminus Y$, $V = X\setminus Z$

$Y = \operatorname{Sing} X$

$$\cdots \longrightarrow \underbrace{H^r_Y(X, \Lambda)}_{\substack{=0 \text{ for } r<2c+2 \\ \text{by induction} \\ (\operatorname{codim} Y \geq c+1)}} \longrightarrow H^r_Z(X, \Lambda) \longrightarrow \underbrace{H^r_{Z\setminus Y}(X\setminus Y, \Lambda)}_{\substack{=0 \text{ for } r<2c \\ \text{by purity } (Z\setminus Y \text{ smooth})}} \longrightarrow \cdots$$

$\Longrightarrow H^r_Z(X,\Lambda) = 0 \ \forall r < 2c.$ □

The above sequence for $r = 2c$ gives the desired isomorphism. This way we have defined the map cl_X, but it is hard to see from here that it has the properties we want it to have. We are going to give an alternative definition. (See [EC, VI.9.2-5] for some more nice properties of the cycle map)

CHERN CLASSES: The idea is to associate to each vector bundle (locally free sheaf of finite rank) a cohomology class $c_* : VB(X) \longrightarrow H^{2*}(X)$ satisfying

(a) Functoriality: $\pi : Y \longrightarrow X$ morphism, $\mathcal{E}$ vector bundle on X

$$c_r(\pi^{-1}\mathcal{E}) = \pi^*(c_r\mathcal{E})$$

(b) Normalization: If $\mathcal{E}$ is a line bundle on X, then $c_0\mathcal{E} = 1$, $c_1\mathcal{E}$ is the image of $\mathcal{E}$ under $\operatorname{Pic} X \longrightarrow H^2(X, \Lambda(1))$ (Kummer)

$\operatorname{Pic} X = H^1(X, \mathbb{G}_m)$, $\Lambda(1) = \mu_n$

(c) Additivity: If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is a SES of vector bundles on X, then $c_t(\mathcal{E}) = c_t\mathcal{E}' \cdot c_t\mathcal{E}''$ (Chern polynomials)

$c_t\mathcal{E} = \sum_{i=0}^{r} c_i\mathcal{E}\, t^i$

These properties characterize the Chern classes uniquely (Grothendieck).

Idea of the construction of the cycle map:

- for each $Z \hookrightarrow X$ consider $\mathcal{O}_Z$ as an $\mathcal{O}_X$-module
- resolve it by vector bundles

$$0 \to \mathcal{E}_n \to \cdots \to \mathcal{E}_0 \to \mathcal{O}_Z \to 0$$

and define $\gamma(Z) = \sum(-1)^i[\mathcal{E}_i]$, $\gamma : C^*(X) \to K_0(VB(X))$

- by additivity, c_* factors through $K_0(VB(X))$ and then

$$C^*(X) \xrightarrow{\gamma} K_0(VB(X)) \xrightarrow{c_*} H^{2*}(X)$$

is the desired map.

- passing to the assoc. graded of K_0 and relabeling shows that we have a functorial graded ring homomorphism. [EC, VI.10.7]. (if $(\dim X - 1)!$ is invertible in n)

Proposition.- [EC, VI.10.6] (Wrong proof) Both constructions coincide

$\mathcal{E}$ v.b on X of rank r $\qquad \mathcal{O}(1)$ $\quad \mathbb{P}(\mathcal{E}) \to X$

$$H^*(X) \to H^*(\mathbb{P}(\mathcal{E})) \qquad 1, \xi, \dots, \xi^{r-1}$$

ξ class of a hyperplane section

$$\begin{cases} c_0\mathcal{E} = 1 \\ \sum_{i=0} \pi_* c_i\mathcal{E}\, \xi^{r-i} = 0 \end{cases}$$

$1 \cdot \xi^r$