Automorphic representations study group

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Abstract

These are (mostly live-TEX-ed) notes from a study group in Oxford from Hilary and Trinity terms 2023. They are quite sketchy, but should give an idea of the material covered. Here's a link to the website.

Contents

Part I

Automorphic representations

1 Algebraic groups and adeles

Speaker: Mick Gielen

In this talk we will mainly introduce a whole bunch of definitions, mostly about adeles, a gadget which packages the information about all the local completions of a number field into one handy topological ring, and algebraic groups, which are the main characters in the Langlands programme. According to the internet, the term "automorphic" was first used by Klein in the 1890s to describe functions, now known as automorphic forms, which are invariant under the action of certain groups. The groups in question will be subgroups of algebraic groups, and an automorphic representation is a representation consisting of automorphic forms.

1.1 Adeles

Definition 1.1. Global fields are finite extensions of $\mathbb{Q}$ or $\mathbb{F}_q(x)$, that is, number fields or function fields.

Definition 1.2. A valuation on a field F is a map $v : F \to \mathbb{R} \sqcup \{\infty\}$ satisfying for all $a, b \in F$,

(i) $v(a) = \infty$ if and only if $a = 0$,

(ii) $v(ab) = v(a) + v(b)$,

(iii) $v(a + b) \geq \min(v(a), v(b))$.

Definition 1.3. An **absolute value** is a function $|\cdot| : F \to \mathbb{R}$ satisfying the usual axioms (see [Getz, def. 1.2], for example). If $0 < \alpha < 1$ and v is any valuation, then $|a|_v := \alpha^{v(a)}$ defines an absolute value on F.

Definition 1.4. Two absolute values are **equivalent** if they induce the same topology; a **place** is an equivalence class of absolute values.

Places correpsonding to non-archmedean absolute values are called **finite**, and the others **infinite**.

Proposition 1.5. *Let F be a global field.*

(i) *If F is a function field, then all places are finite.*

(ii) *If F is a number field, then the infinite places are in bijection with embeddings $F \hookrightarrow \mathbb{C}$ modulo conjugation, and finite places in bijection with prime ideals of $\mathcal{O}_F$. Explicitly, this is given by*

$$\iota : F \hookrightarrow \mathbb{C} \quad \textit{goes to} \quad |x| := |\iota(x)|^{[\iota(F)\otimes\mathbb{R}:\mathbb{R}]}, \tag{1.1}$$

and

$$\mathfrak{p} \leq \mathcal{O}_F \quad \textit{goes to} \quad |x|_{\mathfrak{p}} := q^{-v_{\mathfrak{p}}(x)} \textit{ where } q = \#\mathcal{O}_F/\varpi\mathcal{O}_F \tag{1.2}$$

and $v_{\mathfrak{p}}(x) = \max\{x \in \mathbb{N} : x \in \varpi^n\mathcal{O}_F\}$.

We define completions in the usual way, as equivalence classes of Cauchy sequences with respect to the absolute value.

Definition 1.6. Let F be a global field. We define the **adeles over** F, $\mathbb{A}_F := \prod'_v F_v$, where $\prod'$ denotes the restricted product,

$$\mathbb{A}_F = \{(x_v)_v \in \prod_v F_v : x_n \in \mathcal{O}_{F_v} \text{ for almost all } v\}. \tag{1.3}$$

If v is infinite, we adopt the convention $\mathcal{O}_{F_v} = F_v$. The adeles $\mathbb{A}_F$ has a natural topology generated by fixing a finite set of places S, and for each $v \in S$ fixing $U_v \subset F_v$ open and taking $U = \prod_{v\in S} U_v \times \prod_{v\notin S} \mathcal{O}_{F_v}$.

Proposition 1.7. *$\mathbb{A}_F$ is a locally compact Hausdorff topological ring.*

The diagonal image of F in $\mathbb{A}_F$ is discrete.

Definition 1.8. Let S be a finite set of places. Then

$$\mathbb{A}_F^S := \prod_{v\notin S}{}' F_v \quad \text{and} \quad \mathbb{A}_{F,S} := \prod_{v\in S} F_v. \tag{1.4}$$

We also set $F_\infty = \prod_{v|\infty} F_v$.

Proposition 1.9 (Approximation for adeles)**.** *We have a decomposition $\mathbb{A}_F = F_\infty + \prod_{v\nmid\infty} \mathcal{O}_{F_v} + F$, where we identify F with its diagonally embedded image.*

1.2 Algebraic groups

We are interested in studying algebraic groups like GL_n, SL_n, SO_n etc, which can all be viewed as locally closed subschemes of $\mathrm{Mat}_n \cong \mathbb{A}^{n^2}$ (affine n^2-space, not to be confused with the adeles.)

Example 1.10. We can realise the set $\mathrm{GL}_n(R)$ as the subset of $\mathbb{A}^{n^2+1} = \operatorname{Spec} A$ for $A = R[x_{11}, x_{12} \dots, x_{nn}, y]$ given by $\operatorname{Spec} A/(\det(x_{ij})y - 1)$.

Definition 1.11. An **affine group scheme** is a functor $G : \mathbf{Alg}_F \to \mathbf{Grp}$ represented by an F-algebra, denoted $\mathcal{O}(G)$.

The goal is to use algebrogeometric methods to study matrix groups. A morphism of two affine group schemes is given by a natural transformation of functors, and so we have a category of affine group schemes over F, $\mathbf{AffGrpSch}_F$.

Remark 1.12. We define a morphism $H \to G$ to be injective if $\mathcal{O}(G) \to \mathcal{O}(H)$ is surjective. If F is a field, then this is equivalent to every induced map on F-algebras being injective, but not if F is any ring.

Definition 1.13. G is **linear** if there exists a faithful representation $G \hookrightarrow \mathrm{GL}_n$ for some n.

Definition 1.14. Suppose $F \hookrightarrow F'$ is a field embedding, and G a group scheme over F. Then we define the extension of scalars of G to F' by $G_{F'}(R) := G(R)$.

We can go back as well:

Definition 1.15. $\operatorname{Res}_F^{F'} G(R) := G(R \otimes_F F')$ is called the restriction of scalars.

If F'/F is finite and locally free (as an extension of rings), then the restriction is also linear when G is.

Definition 1.16. An affine algebraic group is a group scheme over F represented by a finitely generated F-algebra.

Proposition 1.17. *Let F be a topological field. Then there is a natural topology on $G(F)$ so that $G(F) \to X(F)$ is continuous for all schemes X/F. This is compatible with imeersions, fibre products etc.*

The following shows that we really only need to care about subgroups of GL_n.

Proposition 1.18. *If G is an algebraic group, then it is linear.*

An element $x \in \mathrm{Mat}_n(\overline{F})$ is *semisimple* if it is diagonalisable over F, *nilpotent* if $x^m = 0$ for some $m \in \mathbb{N}$, and *unipotent* if $x - 1$ is nilpotent.

Similarly, say $x \in G(\overline{F})$ is semisimple (nilpotent, unipotent) if $\phi(x)$ is semisimple (nilpotent, unipotent) for some faithful representation $\phi : G \to \mathrm{GL}_n$. One can check that this does not depend on ϕ.

Theorem 1.19 (Jordan decomposition)**.** *If $x \in G(\overline{F})$, then there exist $x_s, x_u, \in G(\overline{F})$ where x_s is semisimple and x_u is unipotent such that $x = x_s x_u = x_u x_s$.*

Definition 1.20. The **Lie algebra** of G, Lie G, is the kernel of the map

$$G(F[x]/x^2) \to G(F). \tag{1.5}$$

Example 1.21. Let $G = \mathrm{GL}_n$. Then we can find a bijection between Lie G and Mat_n by noting that $(1+\epsilon A)(1-\epsilon A) = 1$, where A is any matrix.

We define a bracket on Lie GL_n by $[X, Y] := XY - YX$, and use this to get brackets on all other linear algebraic groups; note that Lie $G \hookrightarrow$ Lie GL_n.

There is natural action of G on Lie G via conjugation, giving a map $G \to \mathrm{GL}_n(\mathrm{Lie}\, G)$. This is called the **adjoint action**.

We also need the usual algebraic groups $\mathbb{G}_a(A) := (A, +)$ and $\mathbb{G}_m(A) := (A^\times, \times)$.

Definition 1.22. An algebraic group T is called a **torus** if $T_{F^{\mathrm{sep}}} \cong \mathbb{G}_m^r$ for some $r \in \mathbb{N}$, which is called the **rank** of T.

If $T \cong \mathbb{G}_m^r$ without passing to F^{sep}, then T is said to be *split*.

Definition 1.23. A character is an element of $X^*(G) := \mathrm{Hom}(G, \mathbb{G}_m)$.

If $G = T$ is a split torus, then $X^*(T) \cong \mathbb{Z}^r$, but in general it can be smaller. If $X^*(T) = \{0\}$, then T is called **anisotropic**. There is a decomposition $T = T^{\mathrm{anis}} T^{\mathrm{split}}$, where their intersection is finite.

Definition 1.24. The **unipotent radical** of G, $R_u(G)$ is the maximal connected (as scheme) unipotent (all elements are unipotent) normal (closed) subgroup of G.

The radical of a group H is the maximal connected normal solvable subgroup H.

Definition 1.25. If $R(G) = \{1\}$ then G is **semisimple**; if $R_u(G) = \{1\}$, then G is **reductive**.

Note that $R_u(G) \subset R(G)$ so semisimple implies reductive.

Remark 1.26. We are glossing over some details on *smoothness*, which won't be covered here.

Definition 1.27. A **Borel subgroup of** G is a subgroup B such that $B_{F^{\mathrm{sep}}} \subset G_{F^{\mathrm{sep}}}$ is maximal, connected and solvable.

These are nice because G/B is always represented by a projective scheme, and B is minimal with respect to this property.

Definition 1.28. A subgroup P of G is **parabolic** if it contains a Borel subgroup of G, so that G/P is also projective.

Definition 1.29. A torus $T \subset G$ is a **maximal torus** if $T_{F^{\mathrm{sep}}}$ is maximal with respect to inclusion.

Example 1.30. $G = \mathrm{GL}_n$, $T =$ diagonal matrices; this forms a split maximal torus.

Proposition 1.31. *Reductive groups have maximal torii.*

Definition 1.32. We say G is split if a maximal torus is split. If G has a Borel subgroup, then it is quasi-split.

Example 1.33. GL_n has Borel subgroup given by upper triangular (or lower triangular) matrices.

Proposition 1.34 (Levi decomposition)**.** *If $P \subset G$ is a parabolic subgroup, then $P = MN$ where $N = R_u(P)$ and $M \leq P$ is a reductive subgroup.*

2 Hecke algebras and automorphic representations over non-Archimedean fields

Speaker: Zach Feng

Let G be a locally profinite group[1]

Definition 2.1. A rep (π, V) is **smooth** if $\mathrm{Stab}_\pi(v) \subset G$ is open for all $v \in V$. If (π, V) is smooth, then it is **admissible** if $\dim_{\mathbb{C}} V^U < \infty$ for all open subgroups $U \subset G$.

Motivation: Consider $G := \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}) = \prod_p' \mathrm{GL}_n(\mathbb{Q}_p)$, where $\mathrm{GL}_n(\mathbb{Q}_\infty) := \mathrm{GL}_n(\mathbb{R})$. Note that $\mathrm{GL}_n(\mathbb{Q}_p)$ is locally profinite. If $\pi\colon G \to \mathrm{GL}(V)$ is an automorphic representation of G, then $\pi = \bigotimes_p' \pi_p$ where for $p < \infty$, π_p is smooth and admissible.

Goal: (i) show that smooth G-reps are equivalent to modules of certain Hecke algebras, (ii) Spherical Hecke algebras (for GL_n). (iii) Examples for GL_2.

2.1 Hecke algebras

Definition 2.2. Let Ω be a field. An **idempotent algebra** is a pair (H, E) such that H is a (not necessarily unital) Ω-algebra, and $E \subset H$ is a set of idempotents satisfying:

(i) for all $e_1, e_2 \in E$, there exists $e_0 \in E$ such that $e_i e_0 = e_0 e_i = e_i$ for $i = 1, 2$,

(ii) for all $\phi \in H$, there exists $e \in E$ such that $e\phi = \phi e = \phi$.

Definition 2.3. If (H, E) is an idempotent algebra and M an H-module, then for any $e \in E$, define $H[e] := eHe$ and $M[e] := eM$.

Note that $M[e]$ is an $H[e]$-module. We say M is *smooth* if $M = \bigcup_{e\in E} M[e]$, and *admissible* if $\dim_\Omega M[e] < \infty$ for all $e \in E$.

These properties should match up with the corresponding properties for representations.

Definition 2.4. Let $\mathcal{H}$ be a set of compactly supported, locally constant functions $G \to \mathbb{C}$, and for $\phi_1, \phi_2 \in \mathcal{H}$ let

$$(\phi_1 * \phi_2)(g) = \int_G \phi_1(gh^{-1})\phi_2(h)\,dh. \tag{2.1}$$

This makes $\mathcal{H}$ into a $\mathbb{C}$-algebra.

Let K_0 be an open compact subgroup of G, and let

$$\epsilon_{K_0}(g) := \begin{cases} \mathrm{Vol}(K_0)^{-1} & \text{if} \quad g \in K_0 \\ 0 & \quad\text{otherwise.} \end{cases} \tag{2.2}$$

Then ϵ_{K_0} is an idempotent in $\mathcal{H}$. Define $\mathcal{H}_{K_0}$ to be the subalgebra of $\mathcal{H}$ consisting of K_0-bi-invariant functions: $f(kgk') = f(g)$ for all $k, k' \in K_0$ and $g \in G$; a simple computation shows that it is closed under convolution. Moreover, it is unital with identity given by ϵ_{K_0}.

Proposition 2.5. *We have that* $\mathcal{H}_{K_0} = \epsilon_{K_0}\mathcal{H}\epsilon_{K_0}$.

Now let $\phi \in \mathcal{H}_{K_0}$. Then ϕ is constant on sets of the form $K_0 g K_0$ for $g \in G$, we can write $\phi = \sum_g c_g \mathbb{1}_{K_0 g K_0}$ for a finite collection of $g \in G$. For $U \subset G$ open compact, $K_0 = U \cap gUg^{-1}$, we know $K_0 g K_0 \subset gU$, so every neighbourhood of 1 contains a coset of the form $K_0 g K_0$, hence these form a neighbourhood basis.

Accordingly, $\mathcal{H} = \bigcup_{K_0} \mathcal{H}_{K_0}$ so $(\mathcal{H}, \{\epsilon_{K_0}\})$ is an idempotent algebra.

Now let (π, V) be a smooth G-representation. For $\phi \in \mathcal{H}$ and $v \in V$, let $\pi : \mathcal{H} \to \mathrm{GL}(V)$ by $\int_G \phi(g)\pi(g)v\,dg$. Then (π, V) is a smooth $\mathcal{H}$-module: $V = \bigcup_{K_0} \pi(\epsilon_{K_0})V$, and each v is fixed by some K_0.

[1] So, for any nbhd U of the identity there exists an open compact subgroup $K \subset G$ such that $e \in K$ and $K \subset U$

Theorem 2.6. *The category of smooth G-representations is equivalent to the category of smooth $\mathcal{H}$-modules.*

Now fix $K \subset G$ open compact, $\pi(\epsilon_K): V \to V^K$, which is a K-equivariant projection, and V^K is a $\mathcal{H}_K$-module.

Theorem 2.7. *Let (π, V) be an irreducible smooth G-representation, and $K \subset G$ an open compact subgroup.*

(i) *Either $V^K = 0$, or it is simple.*

(ii) *The map sending a smooth irreducible G-rep V with $V^K \neq \{0\}$ to a simple $\mathcal{H}_K$-module V^K, is an bijection.*

Proof of (i). Let $M \subset V^K$ be an $\mathcal{H}_K$-submodule. Then $\pi(\mathcal{H})M$ is G-stable, so $\pi(\mathcal{H})M = V$. Now $V^K = \pi(\epsilon_K)V = \pi(\epsilon_K)\pi(\mathcal{H})M = \pi(\epsilon_K) \dots = M$. We leave the proof of (ii) as an exercise (or a google search). ■

Remark 2.8. If V^K is admissible, then V corresponds to finite-dimensional simple $\mathcal{H}_K$-modules.

Let $G = \mathrm{GL}_n(F)$, $F/\mathbb{Q}_p$ a finite extension with uniformiser ϖ, and $K \coloneqq \mathrm{GL}_n(\mathcal{O}_F)$. Then $\mathcal{H}_K$ is called the *spherical Hecke algebra*.

Theorem 2.9 (p-adic Cartan decomposition)**.** *G has a decomposition*

$$G = \bigsqcup_{e_1 \geq \dots e_n \in \mathbb{Z}} K \begin{pmatrix} \varpi^{e_1} & & \\ & \ddots & \\ & & \varpi^{e_n} \end{pmatrix} K \tag{2.3}$$

Theorem 2.10. *$\mathcal{H}_K$ is commutative.*

Proof. Consider the map $x \mapsto x^t$ in $\mathrm{GL}(F)$, and let σ be the endomorphism of $\mathcal{H}_K$ sending f to $f^\sigma(x) = f(x^t)$. Then, by doing some straightforward substitutions, we find

$$(f_1 * f_2)^\sigma(x) = \int_G f_1(x^t y^{-1}) f_2(y) dy = \int_G f_1^\sigma((y^t x)^{-1}) f_2^\sigma(y^t) dy = \int_G f_1^\sigma(y) f_2^\sigma(xy^{-1}) dy = (f_2^\sigma * f_1^\sigma)(x), \tag{2.4}$$

so we are done if we can show that $\sigma = 1$. But $\mathcal{H}_K$ is spanned as a $\mathbb{C}$-vector space by $\mathbb{1}_{K[\varpi^{e_i}]K}$ by the Cartan decomposition, and these are fixed by σ. ■

Corollary 2.11. *If π is a smooth admissible G-representation, then* $\dim \pi^K \leq 1$.[2]

Proof. π^K is a simple $\mathcal{H}_K$-module, and so is 1-dimensional if it is non-zero. ■

2.2 The case of GL_2

Example 2.12. Let $G = \mathrm{GL}_2(F)$, $K = \mathrm{GL}_2(\mathcal{O}_F)$ as above, fix $B \subset G$ the upper triangular Borel subgroup. Let $\chi_1, \chi_2 : F \to \mathbb{C}^\times$ be characters, and lift to a character on B by $\chi\begin{pmatrix} y_1 & * \\ 0 & y_2 \end{pmatrix} = \chi_1(y_1)\chi_2(y_2)$. Now let

$$\mathcal{B}(\chi_1, \chi_2) = \mathrm{nInd}_B^G \chi \coloneqq \left\{ f : G \to V : f(bg) = |a/d|^{1/2} \chi(b) f(g), \quad \text{and} \quad \exists K_0 \subset G \text{ open cpt s.t. } f(gk_0) = f(g)\ \forall k_0 \in K_0 \right\} \tag{2.5}$$

Proposition 2.13. *The G-representation $\mathcal{B}(\chi_1, \chi_2)$ is irreducible whenever $\chi_1\chi_2^{-1}(u) \neq |y|^{\pm 1}$.*
If χ_1, χ_2 are also unramified (i.e. trivial on $\mathcal{O}_F^\times$), then $\mathcal{B}(\chi_1, \chi_2)^K \neq 0$.

This is called the *normalised induction*, which is nicer than the other because $\mathcal{B}(\chi_1, \chi_2) \cong \mathcal{B}(\chi_2, \chi_1)$. In the last case, write $G = BK$ using the Iwasawa decomposition, so that $\mathcal{B}(\chi_1, \chi_2)^K \mathbb{C}\phi_K$, where $\phi_K(b) = |a/d|^{1/2}\chi_1(a)\chi_2(d)$.
The Hecke algebra has some nice generators for GL_2: $\mathcal{H}_K = \langle T, R, R^{-1} \rangle_{\mathbb{C}}$.

[2] Maybe it's enough to require smooth? But not sure.

Theorem 2.14. *Let $\alpha_i := \chi_i(\varpi)$. Then*

(i) $T\phi_K = q^{1/2}(\alpha_1 + \alpha_2)\phi_K$,

(ii) $R\phi_K = \alpha_1\alpha_2\phi_K$.

Proof. Write $T\phi_K = \lambda\phi_K$, so that

$$\lambda\phi_K(1) = \int_G T(g)\phi_K(g)dg = \int_{K(\ldots)K} \phi_K(g)dg \tag{2.6}$$

and decompose as left cosets .... $= |\varpi|^{1/2}\alpha_2 + q|\varpi|^{1/2}\alpha_2$, which gives (i). ■

If π is an irreducible admissible representation of $\mathrm{GL}_2(F)$, then it is one of the following:

(i) $\mathscr{B}(\chi_1, \chi_2)$ for some χ_i which is irreducible, called the **irreducible principal series**,

(ii) if $\mathscr{B}(\chi_1, \chi_2)$ is reducible, then the Jordan-Hölder decomposition has two factors: a 1-dimensional representation $\chi \circ \det$, and an infinite-dimensional "special" representation,

(iii) If π is not a subquotient of an induced representation, then it is **supercuspidal**.

3 The Satake isomorphism

Speaker: Håvard Damm-Johnsen

References: [Cog04], [Get10, §2],

In this talk, we will introduce the Satake transform, which gives an isomorphism between a Hecke algebra of an algebraic group and a corresponding Hecke algebra of a dual group. We will also try to make this very concrete in the case of GL_2.

3.1 Root systems for algebraic groups

Before continuing the study of Hecke algebras over a local field, we need to review *root systems*, which are fundamental tools in understanding algebraic groups.

Recall from section 1.2 that a *rank r torus T* of an algebraic group G is a subgroup $T \le G$ such that $T \cong \mathbb{G}_m^r$ over an algebraically closed field $\overline{k}$.

Definition 3.1. A **character** of T is a homomorphism $T \to \mathbb{G}_m$. The group $X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m)$ is called the **character group of** T, and its $\mathbb{Z}$-dual $X_*(T) := \mathrm{Hom}(X^*(T), \mathbb{Z})$ is called the **cocharacter group of** T.

Note that since $\mathbb{G}_m$ is abelian, a group homomorphism $\alpha\colon G \to \mathbb{G}_m$ will factor through some torus $T \le G$, so we sometimes call α a character of G.

Exercise 1. *Check that $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$.*

Example 3.2. If $G = \mathrm{GL}_2$, then we have a maximal torus $T = \left\{\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right\}$, and a character $\alpha\colon T \to \mathbb{G}_m$ can be written as $\alpha(t_1, t_2) = t_1^{n_1} t_2^{n_2}$.

Let $\mathfrak{g} := \mathrm{Lie}\, G$. In the first talk we defined the *adjoint action* $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$. Fix a torus $T \le G$ and consider the restriction Ad_T. This gives a commuting family of operators α acting on $\mathfrak{g}$, and we can simultaneously diagonalise these. For each $t \in T$, $\mathrm{Ad}(t) = \alpha(t)$ for some $\alpha \in X^*(T)$, and since each α describes a subspace, there can only be finitely many non-zero α.

Definition 3.3. The non-zero characters α are called **roots of G with respect to T**, and the finite set of non-zero roots is denoted $\Phi(G, T) \subset X^*(T)$.

Proposition 3.4. *Let G be a connected reductive group, and $T \leq G$ a maximal torus with Lie algebra $\mathfrak{t}$ and roots $\Phi = \Phi(G, T)$.*

(i) $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, *where* $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : \mathrm{Ad}(t)x = \alpha(t)x$ *for all* $t \in T\}$.

(ii) *For any* $\alpha \in \Phi$, $T_\alpha := (\ker \alpha)^\circ$ *is a torus of codimension* 1 *in* T.

(iii) *For any* $\alpha \in \Phi$, *there exists a unique* Ad_{T_α}*-stable subgroup* $U_\alpha \leq G$, *and these are permuted by*

$$W(G, T) := N_G(T)/Z_G(T) = \frac{\{g \in G : gTg^{-1} \subset T\}}{\{g \in G : gtg^{-1} = t \text{ for all } t \in T\}}. \tag{3.1}$$

(iv) $G = \langle T, \{U_\alpha \ : \alpha \in \Phi\}\rangle$.

Definition 3.5. The group U_α is called the **root group of α**, and $W(G, T)$ is called the **Weyl group**.

By duality, there is a natural pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$.

Proposition 3.6. *Let $\alpha \in \Phi$ be a root. There exists a unique element $\alpha^\vee \in X_*(T)$ satisfying $\langle \alpha, \alpha^\vee \rangle = 2$.*

Definition 3.7. The map $\alpha^\vee$ is called the **coroot of α**, and the set of (nonzero) coroots is denoted $\Phi^\vee$.

In fact, the map $\alpha \mapsto \alpha^\vee$ is injective, so $\#\Phi = \#\Phi^\vee$.

Definition 3.8. The **root datum of** (G, T) is the tuple $R(G, T) := (X^*, \Phi, X_*, \Phi^\vee)$. The **dual root datum** is $(X_*, \Phi^\vee, X^*, \Phi^\vee)$.

An abstract root datum is a tuple of sets $(X^*, \Phi, X_*, \Phi^\vee)$ satisfying certain axioms found in [Get10, Def. 2.41].

Theorem 3.9 (Chevalley-Demazure). *A connected reductive group over an algebraically closed field is uniquely determined up to isomorphism by its root datum, and any abstract root datum gives rise to a connected reductive algebraic group.*

Note that a root datum determines a root system in the sense of Lie algebras, but contains more information: while a root system determines a semisimple Lie algebra, a root datum will also contain information about the centre, so distinguishes SL_n and GL_n, for example.

Definition 3.10. The **Langlands dual group**, ${}^L G(\mathbb{C})$, is the complex connected reductive group determined by the dual root datum of G. More generally, if G is a reductive algebraic group over a field k, then ${}^L G$ is the group scheme ${}^L G := {}^L(G \times_k \operatorname{Spec} \bar{k}) \rtimes \mathrm{Gal}(\bar{k}/k)$.

G	${}^L G^\circ$
GL_n	$\mathrm{GL}_n(\mathbb{C})$
SL_n	$\mathrm{PGL}_n(\mathbb{C})$
SO_{2n+1}	$\mathrm{Sp}_{2n}(\mathbb{C})$
SO_{2n}	$\mathrm{SO}_{2n}(\mathbb{C})$

Table I.1: Table of split algebraic groups and their duals

3.2 The Satake transform

A good reference here are these notes by James!

Fix a reductive algebraic group G over a local field F, and a compact open $K \le G$. Last week, we defined the *Hecke algebra* $\mathscr{H}_K := C^\infty(K\backslash G/K)$, and explained that a representation (π, V) is *unramified*, or *spherical*, if $V^{K_0} \neq 0$, when $G = \mathrm{GL}_n(F)$ and $K_0 = \mathrm{GL}_n(\mathcal{O}_F)$.

We can do the same for a diagonal torus $T \le G$.

Definition 3.11. A representation $\pi \colon T \to \mathrm{GL}(V)$ is **unramified** if $V^{T_0} \neq 0$, where $T_0 := T \cap K_0$.

If (π, V) is irreducible, then V is one-dimensional, so π is actually a character $\alpha \colon \mathscr{H}(T/T_0) \to \mathbb{C}^\times$. But we can identify $\mathscr{H}(T/T_0)$ with the group of cocharacters $X_*(T)$ via $\mathscr{H}(T/T_0) \cong \mathbb{C}[T/T_0] \leftarrow X_*(T)$ where the last map is $\lambda \mapsto \lambda(\varpi)$. In other words, an irrep is precisely an element of $\mathrm{Hom}_{\mathbb{Z}}(X_*(T), \mathbb{C})$, which is a point on the dual torus $\hat{T}$, of the Langlands dual group ${}^L G$.

Now let N be the unipotent radical of G – for GL_n this could be the upper triangular matrices with 1 along the diagonal – and consider the map $S \colon \mathscr{H}(G, K_0) \to \mathscr{H}(T, T_0)$ defined by

$$S(f)(t) = \delta_B(t)^{1/2} \int_N f(tn) d\mu(n), \tag{3.2}$$

where μ is the Haar measure on N assigning volume 1 to $N \cap K_0$. Here δ_B is the modulus character, i.e. the normalising factor which comes from comparing left and right Haar measures on B, and for GL_2 we have $\delta_B\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = |t_1/t_2|$. One checks that this satisfies

$$S(f)(t) = \delta_B^{-1/2}(t) \int_N f(nt) d\mu(n). \tag{3.3}$$

Definition 3.12. The map S is called the **Satake transform**.

Example 3.13 (GL_2, trivial f)**.** Let's compute the Satake transform of the indicator function of $\mathcal{O}_F$, $\mathbb{1}_{K_0}$. If $t = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ then

$$S(f)(t) = |a/d|^{1/2} \int_N \mathbb{1}_{K_0}(tn) d\mu(n) = |a/d|^{1/2} \mathbb{1}_{T_0}(t) = \mathbb{1}_{T_0}(t). \tag{3.4}$$

In other words, S preserves the unit. It's not much harder to see that it's an algebra homomorphism.

Exercise 2. *Check this, and that the codomain of S is actually $\mathscr{H}(T, T_0)$ as claimed.*

Exercise 3. *Check that $S(K_0\begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}K_0) = q^{1/2}$, where $q := |\varpi|$.*

Example 3.14. Let $\chi \colon T \to \mathbb{C}^\times$ be an unramified character, and consider $I_\chi := \mathrm{nInd}_B^G \chi$. Since $G = BK_0 = TNK_0$ (Iwasawa + Levi decomposition) and χ is unramified, we know that $(I_\chi)^{K_0} = \mathbb{C} \cdot v_\chi$, where v_χ is called a *spherical vector*, acting like $v_\chi(tnk_0) = \delta^{1/2}(t)\chi(t)$.

Now $f \in \mathscr{H}(G, K_0)$ acts on v_χ as a scalar $\pi_\chi(f)$, called the Satake parameter of χ (I think?). By the decomposition above, $\mu_G = \mu_T \times \mu_N \times \mu_{K_0}$, with Haar measures all normalised to give the intersection with K_0 volume 1. We compute

$$\int_G f(g) v_\chi(g) d\mu = \int_T \int_N f(tn) \delta^{1/2} \chi(t) dt dn = \int_T S(f)(t)\chi(t) dt, \tag{3.5}$$

which can be viewed as evaluating $S(f)$ at the point χ in $\hat{T}(\mathbb{C})$.

This is meant to demonstrate that the Satake transform describes the action of the Hecke algebra on unramified principal series representations.

Theorem 3.15. *The Satake transform S is an isomorphism onto the subalgebra $\mathscr{H}(T, T_0)^W$ consisting of functions invariant under the Weyl group W.*

In particular, the structure of $\mathscr{H}(G, K_0)$ is quite simple; it's the invariants of a polynomial ring $\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ under a finite group.

Example 3.16. Let $G = \mathrm{GL}_2$, $F = \mathbb{Q}_p$ and $K_0 = \mathrm{GL}_2(\mathbb{Z}_p)$ as above. Then $G \cong \hat{G}$ (not really, since it's over $\mathbb{C}$, but let's ignore that for now) and $\hat{T} = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} : t_i \in \mathbb{C}^\times \right\}$. The Weyl group is $S_2 = C_2$, and the nontrivial element acts as conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, swapping t_1 and t_2. We find that $\mathscr{H}(G, K_0) = \mathbb{C}[t_1^{\pm} t_2^{\pm}, t_1^{\pm} + t_2^{\pm}]$.

4 Supercuspidals and the local Langlands Correspondence

Speaker: Zach Feng

4.1 Supercuspidals

Let G be a locally profinite group, and (π, V) a smooth representation of G.

Definition 4.1. The **smooth dual of** V is $V^\vee := \bigcup_{K \subset G} (V^*)^K$, where $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$.

Proposition 4.2. *If V is smooth and admissible, then*

(i) *so is $V^\vee$,*

(ii) $V \cong (V^\vee)^\vee$,

(iii) *if V is irreducible, then so is $V^\vee$.*

For $v \in V$, $\lambda \in V^\vee$, define $m_{v,\lambda} \colon G \to \mathbb{C}$ by $m_{v,\lambda}(g) = \lambda(gv)$, the **matrix coefficient of** v **and** λ.

Definition 4.3. A smooth irrreducible representation (π, V) is **supercuspidal** if all matrix coefficients are compactly supported modulo $Z(G)$.

Proposition 4.4. *If V is irreducible, then this is equivalent to checking that a single coefficient is compactly supported modulo $Z(G)$.*

How do we find supercuspidal representations? Let $G = \mathbb{G}(F)$ for some reductive group $\mathbb{G}$, F a local field.

Proposition 4.5. *Let $H \leq G$ be an open subgroup containing Z such that H/Z is compact. Let (σ, W) be a finite dimensional irrep of H. If*

$$\mathrm{cInd}_H^G W := \{f \colon G \to W : f(hg) = \sigma(h) f(g),\ f \text{ cptly supptd. mod } Z\} \tag{4.1}$$

acted on by right translation of G is irreducible and admissible, then it is supercuspidal.

Remark 4.6. It is suspected that all supercuspidals arise these way.

Theorem 4.7 (Fintzen)**.** *If G splits over a tamely ramified extension and $p \nmid \#W$, then all supercuspidals arise this way.*

Proof of proposition 4.5. It suffices to check a single matrix coefficient. By finite-dimensionality of W, we can find $w \in W$, $\lambda \in W^\vee$ such that $\lambda(w) \neq 0$. Define

$$f_w(g) := \begin{cases} \sigma(g) w \text{ if } g \in H, \\ 0 \text{ otherwise,} \end{cases} \quad \text{and} \quad f_\lambda(g) := \begin{cases} \sigma^*(g) w \text{ if } g \in H, \\ 0 \text{ otherwise.} \end{cases} \tag{4.2}$$

Now $\langle f_\lambda, f \rangle := \langle f_\lambda(1), f(1) \rangle$ so that $m_{f_w, f_\lambda}(g) = \langle \lambda, f_w(g) \rangle$. ■

Let $P = MN$ be the Levi decomposition of a proper parabolic P of G, and (π, V) a smooth representation of G. Let $V(N) := \{\pi(n)v - v : v \in V\}$, $V_N := V/V(N)$, and consider M acting on V_N by $\pi|_M \otimes \delta_P^{-1/2}$.

Definition 4.8. The M-module $J_P(V) := V_N$ is called the **Jacquet module** of V wrt P, and sends smooth representations of G to smooth representations of V.

Proposition 4.9. *We have*

(i) *J_P is exact,*

(ii) *J_P preserves admissibility,*

(iii) *J_P is left-adjoint to the parabolic induction functor,* nInd. *In particular, there is a map*

$$\operatorname{Hom}_G(V, \operatorname{nInd}_P^G W) \xrightarrow{\sim} \operatorname{Hom}_G(J_P(V), W) \tag{4.3}$$

Theorem 4.10 (Jacquet). *A smooth admissible irrep (π, V) is supercuspidal if and only if $J_P(V) = 0$ for all proper parabolics $P \leq G$.*

The slogan "supercuspdial means not coming from parabolic induction" is formalised by the following:

Theorem 4.11. *If (π, V) is smooth admissible and irreducible, then there exists a parabolic $P = MN$, and an irreducible supercuspidal (σ, W) of M such that V is a subrepresentation of $\operatorname{nInd}_P^G W$.*

Proof. Since V is irreducible, the statement is equivalent to saying there exists a nontrivial G-equivariant map $V \to \operatorname{nInd}_P^G W$ for some P, W. Proceed by induction on $\dim G$. For $\dim G = 1$, G is a torus, so every function is compactly supported modulo centre. Accordingly, every π is supercuspidal, and $\pi = \operatorname{nInd}_G^G \pi$.

Now suppose $\dim G > 1$. If there are no W, P and G-equivariant map as above, with P proper, then $\operatorname{Hom}(V, \operatorname{nInd}_P^G W) = 0 = \operatorname{Hom}(J_P(V), W)$, so $J_P(V) = 0$, implying V is supercuspidal.

Otherwise, pick a proper parabolic $P \leq G$ and an admissible representation W of M, along with a non-zero map $V \to \operatorname{nInd}_P^G W$. By the adjunction, we get a nontrivial map $J_P(V) \to W$. Because P is proper and $\dim M < \dim G$, we can apply the induction hypothesis to M: there exists a parabolic P' of M with Levi factor M' and supercuspidal W' along with a map $W \to \operatorname{nInd}_{P'}^M W'$. Composing with $J_P(V) \to W$ gives a non-zero map $J_P(V) \to \operatorname{nInd}_{P'}^M W'$, and applying adjunction gives $V \to \operatorname{nInd}_P^G(\operatorname{nInd}_{P'}^M W') = \operatorname{nInd}_{W'N}^G W'$, as required. Finally, by taking an irreducible quotient, we can reduce to the case W' irreducible. ■

Definition 4.12 (Segments). Fix $G = \mathrm{GL}_n(F)$.

(i) For any representation π of G, $s \in \mathbb{Z}$, let $\pi(s) := \pi \otimes |\det|^s$.

(ii) A **segment** is a set of isomorphism classes of irreducible supercuspidal representations of G of the form $\Delta = \{\pi, \pi(1), \dots, \pi(r-1)\} =: [\pi, \pi(r-1)]$.

(iii) Two segments Δ_1 and Δ_2 are **linked** if $\Delta_i \not\subset \Delta_j$, and $\Delta_1 \cup \Delta_2$ is also a segment.

(iv) if $\Delta_1 = [\pi, \pi']$, $\Delta_2 = [\pi'', \pi''']$ are segments, Δ_1 **precedes** Δ_2 if they are linked and $\pi'' = \pi(r)$ for some $r \geq 0$.

Theorem 4.13 (Bernstein–Zelevinsky classification). *Let $P = MN$, with P associated to $n_1 + \dots + n_k = n$ of the form*

(Insert block diagram here) (4.4)

(i) *Let $\sigma = \sigma_1 \otimes \dots \otimes \sigma_k$ where each σ_i is an irreducible supercuspidal. The induction $\operatorname{nInd}_P^G \sigma$ is reducible if and only if there exist $i \neq j$ such that $n_i = n_j$, $\sigma_i = \sigma_j(1)$.*

(ii) *Let $m = n_1 = \ldots = n_k$, so that $n = km$. Then* $\mathrm{nInd}_P^G \Delta := \bigotimes_{i=0}^{k-1} \mathrm{nInd}_P^G \pi(i)$ *has a unique irreducible quotient* $Q(\Delta)$.

(iii) *Consider segments $\{\Delta_i\}_{i=1}^k$ where each $Q(\Delta_i)$ is an irrep of G, and so that Δ_i does not precede Δ_j for any $i < j$.Then* $\mathrm{nInd}_P^G(Q(\Delta_1) \otimes \ldots \otimes Q(\Delta_k))$ *has a unique irreducible quotient denoted* $Q(\Delta_1, \ldots, \Delta_k)$.

Example 4.14. Let $G = \mathrm{GL}_2(F)$, P the upper-triangular Borel, and $P = MN$. Fix characters χ_1, χ_2 of P. Then the irreducible representations of G are:

(i) If $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$, then $\mathrm{nInd}(\chi_1 \otimes \chi_2)$ is irreducible, and $\Delta_i = (\chi_i)$.

(ii) If $\chi_1\chi_2^{-1} = |\cdot|$, let $\chi_2 = \chi|\cdot|^{1/2}$ so that $\chi_2 = \chi|\cdot|^{-1/2}$, giving a short exact sequence

$$0 \to \chi \boxtimes \mathrm{St}_G \to \mathrm{nInd}(\chi_1 \otimes \chi_2) \to \chi \circ \det \to 0, \tag{4.5}$$

and Δ_1 precedes Δ_2; $Q(\Delta_1 \boxtimes \Delta_2)$ is an irreducible quotient of $\mathrm{nInd}(Q(\Delta_1) \otimes Q(\Delta_2))$.

(iii) $\chi_1\chi_2^{-1} = |\cdot|^{-1}$, then there is just one segment, and $\Delta = (\chi|\cdot|^{-1/2}, \chi|\cdot|^{1/2})$ and $Q(\Delta) = \chi \boxtimes \mathrm{St}_G$.

4.2 Local Langlands for GL_n

There exists a unique map $\mathrm{rec}\colon \mathscr{A}_n(F) \to \mathscr{G}_n(F)$ where $\mathscr{A}_n(F)$ is the set of irreducible smooth admissible representations of $\mathrm{GL}_n(F)$, and $\mathscr{G}_n(F)$ is the set of n-dimensional semisimple complex Weil–Deligne representations of the Weil–Deligne group W_F.

This map respects parabolic induction, in the following sense: If $\Delta = [\pi, \pi(r-1)]$, then $\mathrm{rec}(Q(\Delta)) = \mathrm{rec}(\pi) \otimes \mathrm{Sp}(r)$, where $\mathrm{Sp}(n)(z_1, \ldots, z_n) = |z_i|^i z_i$, N is the matrix with 1 on the superdiagonal and 0 elsewhere.[3] Here $\mathrm{Sp}(n)$ is the image of the Steinberg representation in the category of the Weil–Deligne representations. Moreover, $\mathrm{rec}(Q(\Delta_1) \boxplus \ldots \boxplus Q(\Delta_k)) = \bigoplus_{i=1}^k \mathrm{rec}(Q(\Delta_i))$.

The representations from the previous section have the following images:

(i) $(\chi_1 \oplus \chi_2, 0 \otimes 1 + 1 \otimes 0)$

(ii) $(\chi_1|\cdot|^{1/2} \oplus \chi_1|\cdot|^{-1/2}, 0)$,

(iii) $(\chi_1|\cdot|^{-1/2} \oplus \chi_1|\cdot|^{1/2}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$.

5 Representations of real reductive groups

Speaker: James Newton

Let $\mathbf{G}$ be a connected reductive group over $\mathbb{R}$; we want to study real representations of $G := \mathbf{G}(\mathbb{R})$. Let $K \subset G$ be a maximal compact subgroup. It turns out that $K = \mathbf{K}(\mathbb{R})$ for some algebraic subgroup $\mathbf{K} \subset \mathbf{G}$.

Zhiwei Yun's notes are a good reference for this talk.

The key example to keep in mind is the following:

Example 5.1. $\mathbf{G} = \mathrm{GL}_n/\mathbb{R}$ and $K = O(n)$.

Also of interest is the following:

Example 5.2. Let $\mathbf{G} = \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}} \mathrm{GL}_n/\mathbb{C}$; $G = \mathrm{GL}_n(\mathbb{C})$ and $K = U(n)$.

For a less trivial example, we can study representations $U(p,q)$, the unitary groups of mixed signature. These are not always quasi-split!

[3]Tensor product is defined differently on RHS!

5.1 Hilbert space representations

Let V be a Hilbert space with a continuous action of G, meaning $G \times V \to V$ is a continuous map.

Definition 5.3. A representation of G is **unitary** if G is norm-preserving, $\|gv\| = \|v\|$ for all $v \in V$ and $g \in G$.

Classical problem: classify unitary representations of G. A key person in the resolution of this was Harish-Chandra, a student of Dirac.

Less classically, one might think of how $L^2(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}))$ decomposes under the action of $\mathrm{SL}_2(\mathbb{R})$ by right translation. More generally, we can consider an adelic quotient $L^2_\psi(\mathrm{GL}_n(\mathbb{Q}) \backslash \mathrm{GL}_n(\mathbb{A}_\mathbb{Q}))$; here $\psi\colon \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ is a unitary central character, and for $f \in L^2$, we require $f(gz) = f(g)\psi(z)$ for all $z \in \mathbb{A}_\mathbb{Q}^\times$.

This gives a unitary representation of $\mathrm{GL}_n(\mathbb{R})$, and automorphic forms show up in these.

Harish-Chandra had the idea that instead of looking at this horribly big space alone, we can look at how the Lie algebra acts. Let $\mathfrak{g} := (\mathrm{Lie}\, G)_\mathbb{C}$, and let $g_\mathbb{R} := \mathrm{Lie}\, G$.

Definition 5.4. Let V be a Hilbert space representation of G. Then v is **differentiable** if for all $X \in \mathfrak{g}_\mathbb{R}$, the limit

$$X \cdot v := \lim_{t \to 0} \frac{\exp(tX)v - v}{t} \tag{5.1}$$

exists in V. It is **smooth** if we can iterate this indefinitely.

Now we can extend by linearity to get an action of $\mathfrak{g}$ and $U(\mathfrak{g})$, the universal enveloping algebra. Let V^∞ denote the space of smooth vectors.

Exercise 4. *Check that V^∞ is G-stable.*

Theorem 5.5 (Gårding)**.** *V^∞ is dense in V.*

Another way to "cut down the representation" is by looking at the action of the compact subgroup K.

Definition 5.6. A vector $v \in V$ is **K-finite** if it is contained in a finite-dimensional K-stable subspace of V.

In other words, if $\{k \cdot v : k \in K\}$ spans a finite-dimensional subspace of V. The space of K-finite vectors is denoted by $V^{K-\mathrm{fin}}$.

Note that by using Weyl's averaging trick, $V|_K$ is unitarisable, so $V|_K = \hat{\bigoplus}_{\sigma \in \mathrm{Irr}(K)} V(\sigma)$ where $V(\sigma) := \sigma \otimes \mathrm{Hom}_K(\sigma, V) \hookrightarrow V$ and each σ is finite-dimensional. From this we deduce that $V^{K-\mathrm{fin}} = \bigoplus_\sigma V(\sigma)$.

Definition 5.7. V is **admissible** if $V(\sigma)$ is finite-dimensional for all σ.

One thing we've lost is that $V^{K-\mathrm{fin}}$ is not G-stable in general.

Proposition 5.8. *If V is admissible, then $V^{K-\mathrm{fin}} \subset V^\infty$ and stable under the action of $U(\mathfrak{g})$.*

This shows that an admissible V gives a prototypical example of a $(\mathfrak{g}, K)$-module, which we now define:

5.2 $(\mathfrak{g}, K)$-modules

Definition 5.9. A **$(\mathfrak{g}, K)$-module** is a complex vector space[4] V with "nice" compatible actions of $\mathfrak{g}$ and K, meaning:

(i) V is K-finite, and for any $v \in V$, the action of K on any finite-dimensional subspace $V_0 \ni v$ is continuous.[5]

(ii) $\mathfrak{k} := \mathrm{Lie}\, K$ acts on V, coinciding with the $\mathfrak{g}$-action (after extending to $\mathbb{C}$)

[4] No topology specified!

[5] By the Peter-Weyl theorem, this implies K acts smoothly.

(iii) if $k \in K$ and $X \in \mathfrak{g}$, then $k \cdot (Xv) = (\mathrm{Ad}(k)X)(kv)$.[6]

Definition 5.10. Let V be a $(\mathfrak{g}, K)$-module. Then V is **admissible** if $V(\sigma)$ is finite-dimensional for any irrep. σ of K.

In particular, if V is an admissible Hilbert space rep. of G, then $V^{K-\text{fin}}$ is an admissible $(\mathfrak{g}, K)$-module.

There is a bijection between closed G-subrepresentations of V and sub-$(\mathfrak{g}, K)$-modules of $V^{K-\text{fin}}$. Thus, admissible topologically irreducible G-representations map to irreducible $(\mathfrak{g}, K)$-modules (but is not injective; two representations with same $(\mathfrak{g}, K)$-modules are called *infinitesimally equivalent*). Our goal will eventually be to classify the latter; this gives rise to the Langlands classification, somewhat analogous to the Bernstein–Zelevinsky classification.

Another interesting question is: which $(\mathfrak{g}, K)$-modules come from unitary representations? That is, which modules are unitarisable? This is apparently not fully understood for all G.

Proposition 5.11 (Schur's lemma for $(\mathfrak{g}, K)$-modules)**.** *For an irreducible admissible $(\mathfrak{g}, K)$-module V, $\mathrm{End}(V) = \mathbb{C}$.*

Let $Z(\mathfrak{g})$ denote the centre of $U(\mathfrak{g})$. Then $Z(\mathfrak{g})$ acts via a character $\lambda \colon Z(\mathfrak{g}) \to \mathbb{C}$, called the *infinitesimal character of V*.

Let $\mathfrak{h} \subset \mathfrak{g}$ denote the Cartan subalgebra. Then $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Then (fact!) $Z(\mathfrak{g}) \subset U(\mathfrak{h}) \oplus U(\mathfrak{g})\mathfrak{n}^+ \xrightarrow{\text{pr}} U(\mathfrak{h})$.

Let $\rho = 1/2 \sum_{\partial \in R_+} \partial$ be the sum of the weights of $\mathfrak{h}$ acting on $\mathfrak{n}^+$, and consider the "twist" $t : h \mapsto h - \rho(h)1$.

Theorem 5.12 (Harish-Chandra)**.** *The composite $t \circ \mathrm{pr}$ and the inclusion of $U(\mathfrak{g})$ gives a canonical*[7] *isomorphism* $\mathrm{HC} \colon Z(\mathfrak{g}) \to U(\mathfrak{h})^W$.

This can be seen as an archimedean version of the Satake isomorphism. Therefore, we can think of an infinitesimal character $\lambda \colon Z(\mathfrak{g}) \to \mathbb{C}$ as a W-orbit of characters $\mathfrak{h} \to \mathbb{C}$.

Example 5.13. For $G = \mathrm{GL}_2(\mathbb{R})$, $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{C})$, then $\mathfrak{n}^+$ is the upper triangular (resp. $\mathfrak{n}^-$ the lower triangular matrices), $\mathfrak{h}$ the diagonal matrices. Then $Z(\mathfrak{g}) = \mathbb{C}[z, \Delta]$ where Δ is the Casimir element and $z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; pick standard basis $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then

$$\Delta = \frac{1}{2}h^2 + fe + ef = \frac{1}{2}h^2 + h + 2fe \in U(\mathfrak{h}) \oplus U(\mathfrak{g})\mathfrak{n}^+. \tag{5.2}$$

Then $\mathrm{pr}(\Delta) = \frac{1}{2}h^2 + h$, $W = \langle \sigma \rangle \cong C_2$, $\sigma h = -h$. Now $\mathrm{pr}(\Delta)$ is not W-invariant, but twisting sends h to $h - 1$ so $\mathrm{HC}(\Delta) = \frac{1}{2}(h-1)^2 + h - 1 = \frac{h^2-1}{2}$ which is an even polynomial, hence invariant under $h \mapsto -h$, which is exactly the action of W on $\mathfrak{h}$.

5.3 Classification of irreducible $(\mathfrak{g}, K)$-modules

Slogan: "Everything is a submodule of a parabolic induction from a minimal parabolic". [8]

Let's restrict our attention to $\mathrm{SL}_2(\mathbb{R})$.[9] Let B be the upper-triangular Borel (a minimal parabolic); the *Langlands decomposition* is given by $B = MAN$ where $M = \{\pm I\}$, $A = \left\{\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\right\}$, $a > 0$, and $N = \left\{\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}\right\}$. A representation of B is determined by $\epsilon \in \{0, 1\}$ "mod 2" determining $\pm I \mapsto (\pm 1)^\epsilon$, $\lambda \in \mathbb{C}$, determining $a \mapsto a^\lambda$, and setting it trivial on N. Then $\left(\mathrm{Ind}_B^G \epsilon \otimes (\lambda + 1)\right)^{K-\text{fin}}$ is defined to be the *principal series* $V(\epsilon, \lambda)$, which is a $(\mathfrak{g}, K)$-module with infinitesimal character $h \mapsto \lambda$. These will be the basic building blocks:

[6] This follows from (ii) when K is a connected Lie group.

[7] Meaning, not dependent on the choice of $\mathfrak{n}^+$!

[8] So unlike the non-archimedean case, there are no supercuspidals!

[9] $GL_2(\mathbb{R})$ is cleaner to state, but messier to set up; see Yun's notes.

Theorem 5.14. *$V(\epsilon,\lambda)$ is irreducible unless:*

(i) $\lambda \in \mathbb{Z}$, $\epsilon \equiv \lambda+1 \bmod 2$, *in which case* $\mathrm{Sym}^n\,\mathbb{C}^2$ *is a subquotient of* $V(n \bmod 2, -n-1)$ *and* $V(n \bmod 2, n+1)$, *and the other JH-constituent is a sum of the* holomorphic and anti-holomorphic discrete series,

(ii) $V(1,0)$, *called the* limit of discrete series.

When $\lambda \in i\mathbb{R}$, then $V(\epsilon,\lambda)$ is unitary; for $n=0$, $\mathbb{C}$ the trivial rep of $\mathrm{SL}_2(\mathbb{R})$ is unitary. The discrete series and limit of discrete series are unitary, but to prove this, one needs to find a different realisation of the representations.

The *complementary series* are given by $V(\epsilon,\lambda)$, $\epsilon=0$, $\lambda\in(-1,1)$, and these are also unitarisable!

Holomorphic modular forms can be seen as vectors in the holomorphic discrete series. Maaß forms give vectors in either (i) the principal series $V(\epsilon,\lambda)$ for $\lambda \in i\mathbb{R}$, and in particular certain eigenvalue $1/4$ forms correspond to "algebraic" Maaß forms, or (ii) limits of discrete series $V(1,0)$. Maass forms with Laplace-Beltrami eigenvalue in $(0,1/4)$ appear in the complementary series, but Selberg's $1/4$-conjecture states that no such exist!

6 Automorphic representations

Speaker: Alex Horawa

Let G be a connected reductive group over a number field F, Σ the set of places of F, Σ_∞ the subset of infinite places. If $S\subset\Sigma$, then $\mathbb{A}_F^S$ is the adeles which are 0 in the components at places in S, and $\mathbb{A}_{F,S}$ its complement in $\mathbb{A}$.

Recall: $L^2_\psi(G(F)\backslash G(\mathbb{A}_F))$ is an admissible $(\mathfrak{g},K)$-module with an additional action of $G(\mathbb{A}_F)$; we write $\mathbb{A}_F^\infty$ for the adeles away from infinity. Then the K-finite vectors of the space are a $(\mathfrak{g},K)\times G(\mathbb{A}_F^\infty)$-module.

Definition 6.1. An **automorphic representations** is an admissible $(\mathfrak{g},K)\times G(\mathbb{A}_F^\infty)$-module isomorphic to an irreducible subquotient of the K-finite vectors of $L^2_\psi(G(F)\backslash G(\mathbb{A}_F))$.

6.1 Representations and Hecke algebras

We give a quick recap of section 2.1. Let $C_c^\infty(G(\mathbb{A}_{F,S}))$ be the set of locally constant functions on $G(\mathbb{A}_{F,S})$. Let $\mathscr{H}^S$ be the Hecke algebra away from S, meaning the component of places in S is the indicator function of $\mathcal{O}_{F_p}$, or constantly 1

Definition 6.2. A representation (π,V) of $\mathscr{H}^\infty$ is **admissible** if for all $K^\infty \le G(\mathbb{A}_F^\infty)$ open compact, $\pi^{K^\infty}=\pi(\mathbb{1}_{K^\infty})V$ is finite-dimensional and it is *non-degenerate*[10].

We can also define a Hecke algebra at ∞ as follows: $G(\mathbb{R}) := (\mathrm{Res}^F_{\mathbb{Q}}\,\mathbb{G})_{\mathbb{R}}$, $\mathbb{G}(\mathbb{R}) = G(F\otimes_{\mathbb{Q}}\mathbb{R})$ is a real reductive group over $\mathbb{R}$, and fix $K_\infty \le G(\mathbb{R})$.

Definition 6.3. The Hecke algebra at ∞ is $\mathscr{H}_\infty := \mathscr{H}(G(\mathbb{R}),K_\infty)$ the convolution algebra of distributions of $G(\mathbb{R})$ supported at K_∞.

Given a representation $K_\infty \to \mathrm{Aut}(V)$ of dimension n, we get a central character and a Haar measure on K_∞, dK_∞, and can define an idempotent

$$\mathbb{1}_\infty = ? \tag{6.1}$$

Definition 6.4. A continuous representation of K_∞ on a Hilbert space V is **admissible** if V is an irreducible representation of K_∞ and $\pi(1_\infty)V$ is finite-dimensional.

Like in the p-adic case, a $(\mathfrak{g},K)$-rep is admissible if and only if the associated $\mathscr{H}_\infty$-representation is admissible.

Definition 6.5. The **global Hecke algebra** of G is defined to be $\mathscr{H} := \mathscr{H}^\infty \otimes \mathscr{H}_\infty$.

[10]Technical condition I didn't quite get

By definition, a representation (π, V) of $\mathscr{H}$ corresponds to a product $(\pi^\infty, V^\infty) \boxtimes (\pi_\infty, V_\infty)$.

Definition 6.6. (π, V) is **admissible** if π^∞ and π_∞ are both admissible.

Note that there is a natural action of $\mathscr{H}$ on $L^2_\psi(G(F)\backslash G(\mathbb{A}_F))$ by convolution. We can use this to get an alternative, equivalent definition of an automorphic representation:

Definition 6.7 (V2)**.** An **automorphic representation** is an $\mathscr{H}$-representation which is isomorphic to a subquotient of $L^2_\psi(G(F)\backslash G(\mathbb{A}_F))$.

This point of view is more useful to prove the main theorem of the next section.

6.2 Flath's factorisation theorem

Theorem 6.8 (Flath's theorem)**.** *Let π be an automorphic representation. Then for each $v \in \Sigma$ there exists a representation π_v of $G(F_v)$, such that*

$$\pi \cong \bigotimes_v{}' \pi_v \tag{6.2}$$

We ought to explain what the prime in the tensor product means; the following is probably not quite correct.

Definition 6.9. Let I be a countable index set, $I_S \subset I$ a finite subset, and $\{V_v\}_{v \in I}$ a collection of $\mathbb{C}$-vector spaces. Let $\phi_v \in V_v$ be a fixed vector. Then

$$W = \bigotimes_{v \in V}{}' := \{(w_v)_v : w_v = \phi_v \text{ for a.e. } v \in I\}. \tag{6.3}$$

Will define $\bigotimes_v{}' V_v = \lim_{S \text{ fin}} \bigotimes_{v \in S} V_v$ for vector spaces. To make this compatible with the algebra structure, we need the transition maps to be something like $\phi \mapsto \phi \otimes \phi^0_{v_1} \otimes \dots \phi^0_{v_n}$ where $\phi^0_{v_i}$ are idempotents.

Now the central question is, how does this decomposition interact with group actions?

Example 6.10. Take $\mathscr{H} = \mathscr{H}^\infty \oplus \mathscr{H}_\infty$. Then $\mathscr{H}^\infty$ is the restricted product of $\mathscr{H}_v$ with respect to the standard idempotents $\mathbb{1}_{K_v}$ where K_v is a *hyperspecial subgroup*[11], for example $G(\mathscr{O}_F)$.

Definition 6.11. A $C^\infty_c(G(\mathbb{A}^\infty_F))$-module W is **factorisable** if W is irreducible and $W \cong \bigoplus_v{}' W_v$ with $\phi_v \in W_v^{K_v}$, where each $W_v^{K_v}$ is 1-dimensional.

If this is the case, then up to rescaling the choice of compatible system is irrelevant.

Theorem 6.12 (Flath's theorem (V2))**.** *If W is an admissible irreducible representation of $\mathscr{H}^\infty$, then W is factorisable.*

Proof. **Step 1.** "Weak version":

Proposition 6.13. *Let G_1 and G_2 be locally profinite groups, $G = G_1 \times G_2$.*

(i) *If V_i is an admissible irreducible representation of G_i, then $V_1 \otimes V_2$ is an admissible irreducible representation of G.*

(ii) *If V is an admissible irrep of G, then there exist irreps V_1 and V_2 such that $V \cong V_1 \oplus V_2$, and the isomorphism class of each V_i is determined by V.*

Proof idea. Because V is smooth, we can reduce it to a statement about Hecke algebras with respect to compact opens, which behave nicely with respect to products. ■

Step 2. The yoga of Gelfand pairs implies:

[11] Alex doesn't know what this means, so I don't need to either.

Proposition 6.14. *Suppose G is unramified outside S, for each $v \notin S$, $K_v := G(\mathcal{O}_v)$ and $K^S = \prod_{v \notin S} K_v$.*

If V^S is irreducible and admissible then $\dim V^{K^S} = 1$.

Now let W be an irreducible admissible representation of $C_c^\infty(G(\mathbb{A}_F)//G(F)) = \bigotimes_v{}' \mathcal{H}_v$ (or whatever).

Consider W^{K_S} is an A_S-representation where A_S = product of Hecke algebras away from S. Then $W^{K^S} = \bigotimes_{v \in S} W_1 \otimes W^S$ by the first proposition, and by the second, $\dim W^S = 1$.

Our goal is to show that $W = \operatorname{proj\,lim} W^{K_S}$ where $W^{K_S} = \bigotimes_{v \in S} W_v \otimes W^S$, where W^S is the spherical vectors. (?) ■

6.3 Automorphic multiplicity

Let (π, V) be an AIR, and pick π_v's as in Flath's theorem; suppose further that it is unramified at only finitely many places. When is π an automorphic representation? More generally, what is the multiplicity of π in $L^2_\psi(G(\mathbb{A}_F)/G(F))$?[12]

Definition 6.15. An element $\phi \in L^2(G(F)\backslash G(\mathbb{A}))$ is **cuspidal** if for any parabolic $P = MN$ we have

$$\int_{N(F)\backslash N(\mathbb{A})} \phi(ng)dn = 0. \tag{6.4}$$

The linear subspace of L^2 consisting of cupsidal automorphic representations is denoted $L^2_{\text{cusp}}(G(F)\backslash G(\mathbb{A}))$.

Definition 6.16. Let π be an admissible irreducible representation of $G(\mathbb{A})$.

(i) The **multiplicity of** π is $m(\pi) := \dim \operatorname{Hom}_{G(\mathbb{A})}(\pi, L^2_{\text{cusp}} G(F)\backslash G(\mathbb{A}))$.

(ii) We say π is **equivalent** to π' if $\pi \cong \pi'$ as $G(\mathbb{A})$-representaitons.

(iii) π and π' are **weakly equivalent** if $\pi_v \cong \pi'_v$ for almost all places v of F.

Theorem 6.17 (Piatetski-Shapiro). *Let π be an automorphic representation of* $\mathrm{GL}_n(\mathbb{A})$.

(i) *(Multiplicity one) $m(\pi) = 1$: if π and π' are equivalent, then $\pi = \pi'$.*

(ii) *(Strong multiplicity one) if π and π' are weakly equivalent, then they are isomorphic.*

We will see in the next lecture that the second statement generalises the statement that modular eigenform is uniquely determined by a cofinite set of Hecke eigenvalues.

7 Modular forms and automorphic representations

Speaker: Arun Soor[13]

In this talk, we will describe how modular forms naturally give rise to automorphic representations, and how to go back. For the rest of the section, let:

(i) $G = \mathrm{GL}_2/\mathbb{Q}$

(ii) $K_\infty \leq G(\mathbb{R})$ the maximal compact subgroup, isomorphic to $O(2)$,

(iii) $Z_\infty \leq G(\mathbb{R})$ the centre of $G(\mathbb{R})$, isomorphic to $\mathbb{R}^\times \cong \mathbb{R}^\times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

[12] At this point, Zach's computer ran out of battery, so the rest is a reconstruction from Alex's notes.

[13] Disclaimer: these notes are an "expanded version" of the talk, including a lot more words than Arun said. For his (terser) notes, see the website.

Let $G(\mathbb{R})^+$ be the connected component of the identity in $G(\mathbb{R})$, or equivalently, the set of matrices with positive determinant. For any subgroup $H \leq G(\mathbb{R})$, let $H^+ = H \cap G(\mathbb{R})^+$.

Exercise 5. *Show that $O(n)$ is the maximal compact subgroup by $\mathrm{GL}_n(\mathbb{R})$, by noting that $O(n)$ is compact, that any compact subgroup of $G(\mathbb{R})$ fixes some inner product, hence is conjugate to $O(n)$, and finally that $gO(n)g^{-1}$ has the same dimension and number of connected components as $O(n)$.*

The group $G(\mathbb{R})^+$ acts on $\mathfrak{h} := \{z \in \mathbb{C} : \Im z > 0\}$ by linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}, \tag{7.1}$$

and this is "almost free"; the only point with a non-trivial stabiliser is i.

Exercise 6. *Show that* $\mathrm{Stab}\, i = Z_\infty^+ K_\infty^+$, *where*

$$Z_\infty = \left\{ z\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : z \in \mathbb{R} \right\} \quad \text{and} \quad K_\infty^+ = \mathrm{SO}_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : 0 \leq \theta < 2\pi \right\}. \tag{7.2}$$

It follows that we have an equality of sets $\mathfrak{h} = G(\mathbb{R})^+ / Z_\infty^+ K_\infty^+$. From this, it is not a stretch to imagine that we can reinterpret modular forms as functions on $G(\mathbb{R})$.

7.1 Modular forms as automorphic forms

Let $S_k(N, \chi)$ denote the modular cusp forms of weight k, level

$$\Gamma_1(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\} \tag{7.3}$$

and Nebentypus $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$; in other words, the $\mathbb{C}$-vector space of holomorphic functions $f\colon \mathfrak{h} \to \mathbb{C}$ satisfying

$$f(\gamma z) = \chi(d)(cz+d)^k f(z) \quad \text{for all} \quad \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_0(N), \tag{7.4}$$

and which tend to 0 as $z \to i\infty$. Writing $f|_k\gamma(z) := (cz+d)^{-k} f(\gamma z)$, eq. (7.7) simply says $f|_k\gamma = \chi(d) f$.

Theorem 7.1 (Strong approximation for SL_2)**.** *For any open subgroup $U \leq \mathrm{SL}_2(\mathbb{A}^\infty)$, we have $\mathrm{SL}_2(\mathbb{A}) = \mathrm{SL}_2(\mathbb{Q})\,\mathrm{SL}_2(\mathbb{R}) U$.*

One reference for this is the appendix in [Gar90] (email me for a pdf!).

The corresponding statement for GL_1 is that $\mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}_+^\times \hat{\mathbb{Z}}^\times$; this is equivalent to the Chinese remainder theorem.

By combining the two, we get:

Theorem 7.2 (Strong approximation for GL_2)**.** *For any open compact $K^\infty \subset G(\mathbb{A}^\infty)$ such that $\det(K^\infty) = \hat{\mathbb{Z}}^\times$ we have $G(\mathbb{A}) = G(\mathbb{Q}) G(\mathbb{R})^+ K^\infty$.*

It follows that

$$G(\mathbb{Q}) \backslash G(\mathbb{A}) / K^\infty \cong \Gamma \backslash G(\mathbb{R})^+ \tag{7.5}$$

where Γ is the image of $G(\mathbb{Q}) \cap G(\mathbb{R})^+ K^\infty$ in $G(\mathbb{R})^+$. To be completely explicit, the group $G(\mathbb{R})^+$ acts on $G(\mathbb{Q}) \backslash G(\mathbb{A}) / K^\infty$ by right multiplication in the ∞-component, and $G(\mathbb{Q}) g K^\infty g_\infty = G(\mathbb{Q}) g K^\infty g'_\infty$ if and only if $g'_\infty g_\infty^{-1} \in G(\mathbb{Q}) \cap K^\infty$. This shows that the map $\Gamma g_\infty \mapsto G(\mathbb{Q})(1, \dots, g_\infty) K^\infty$ is injective[14], and it is surjective by theorem 7.2.

[14] Further details can be found in Jeremy Booher's notes

Example 7.3. For

$$K^\infty = K_0(N) := \left\{\gamma \in G(\mathbb{A}^\infty) \cong G(\hat{\mathbb{Z}}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N\right\} \tag{7.6}$$

we get $\Gamma = \Gamma_0(N)$. We can similarly define $K_1(N)$.

For f a modular form as above, let $\phi_f \in G(\mathbb{R})^+ \to \mathbb{C}$ be the function

$$\phi_f(g_\infty) = f(g_\infty i) j(g_\infty, i)^{-k} \quad \text{for} \quad g_\infty = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G(\mathbb{R})^+ \tag{7.7}$$

where $j(g_\infty, i) := \det(g_\infty)^{-1/2}(cz + d)$.[15]

Exercise 7. *Check that $j(g_\infty, z)$ satisfies*

$$j(g_\infty g'_\infty, z) = j(g_\infty, g'_\infty z) j(g'_\infty, z) \quad \text{and} \quad j(z_\infty k_\theta, i) = \operatorname{sgn}(z_\infty) e^{i\theta}, \tag{7.8}$$

for $z \in \mathfrak{h}$, $g_\infty, g'_\infty \in G(\mathbb{R})$, $z_\infty \in Z_\infty$ and $k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in K_\infty^+$.

Exercise 8. *Extend the slash operator to $g_\infty \in G(\mathbb{R})^+$ by $f|_k g_\infty(z) = j(g_\infty, z)^{-k} f(z)$. Then $\phi_f(g_\infty) = (f|_k g_\infty)(i)$.*

Note that we can recover the value of f at $z = x + iy$ from ϕ_f by evaluating at

$$g_z := \begin{pmatrix} y^{1/2} & xy^{1/2} \\ 0 & y^{-1/2} \end{pmatrix} : \tag{7.9}$$

indeed, $g_z \cdot i = z$, so $\phi_f(g_z) = f|_k g_z(i) = f(z) y^{k/2}$.

We get two transformation laws for ϕ_f: one for the level ("finite data"):

$$\phi_f(\gamma g_\infty) = \phi_f(g_\infty)\chi(d) \quad \text{for} \quad \gamma = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in \Gamma_0(N), \tag{7.10}$$

and one for the weight ("infinity data"):

$$\phi_f(g_\infty z_\infty k_\theta) = \phi_f(g_\infty)(\operatorname{sgn} z_\infty)^k e^{-ik\theta}. \tag{7.11}$$

We want to use eq. (7.5) to turn ϕ_f into a function on $G(\mathbb{A})$. The Nebentypus character χ only records data "at infinity", but we will use it to define a Hecke character (i.e. a continuous character on $\mathbb{Q}^\times \backslash \mathbb{A}^\times$) ω as follows:

(i) ω is trivial on $\mathbb{Q}^\times \mathbb{R}_{>0}$.

(ii) $\omega(d) = \prod_{p|N} \omega_p(d_p)$, when $d = (d_p)_p \in \mathbb{A}^\infty$.

(iii) $\omega_p(d_p) \equiv \chi(d_p)^{-1} \bmod p^{\operatorname{ord}_p N}$.

(iv) Let π_p be the image of p under $\mathbb{Q}_p^\times \hookrightarrow \mathbb{A}^\times$, so that $\pi_p = p \cdot 1 \cdot \alpha$ for $\alpha = \pi_p/p$ in the decomposition $\mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}^\times \hat{\mathbb{Z}}$. Then $\omega(\pi_p) = \chi(1/p)^{-1} = \chi(p)$.

A less concrete way to define ω is to identify $\mathbb{Z}/N\mathbb{Z}$ with $\hat{\mathbb{Z}}/N\hat{\mathbb{Z}}$, and lifting to a character on $\hat{\mathbb{Z}}$.

Now we do the usual thing for Nebentype characters: extend to $K_0(N)$ by

$$\omega\begin{pmatrix} * & * \\ * & d \end{pmatrix} := \omega(d) \quad \text{for} \quad \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in K_0(N). \tag{7.12}$$

Note that this is multiplicative.

We can now extend ϕ_f to $G(\mathbb{A})$ as follows:

[15] This is the convention from [Gel73], but conventions vary!

Definition 7.4. Fix $f \in S_k(N,\chi)$. The **automorphic form attached to** f is the function

$$\phi_f \colon G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C} \tag{7.13}$$

defined by $\phi_f(g) := \phi_f(g_\infty)\omega(k) = (f|_k g_\infty)(i)\omega(k)$ where by theorem 7.2, $g \in G(\mathbb{A})$ is written $g = \gamma g_\infty k$ for $\gamma \in G(\mathbb{Q})$, $g_\infty \in G(\mathbb{R})$ and $k \in K_0(N)$.

As before, we can recover f by evaluating ϕ_f at the tuple $(1,\dots,g_z)$ with g_z defined in eq. (7.9).

Exercise 9. *Using the decomposition* $\mathbb{A}^\times = \mathbb{Q}^\times\mathbb{R}^\times\hat{\mathbb{Z}}$, *check that* $\phi_f(zg) = \omega(z)\phi_f(g)$ *for all* $g \in G(\mathbb{A})$.

This justifies calling ω the **central character of** f. Similarly, we can encode the weight of ϕ_f by the character $\sigma_k \colon K_\infty^+ \to \mathbb{C}^\times$ defined as $\sigma_k(k_\theta) = e^{-ik\theta}$.

Exercise 10. *Check that eq. (7.11) is equivalent to* $\phi_f(gk_\theta) = \phi_f(g)\sigma_k(k_\theta)$ *for all* $g \in G(\mathbb{A})$ *and* $k_\theta \in K_\infty^+$.

Remark 7.5. If we view $G(\mathbb{A})$ as acting on ϕ_f by right multiplication (meaning $g \cdot \phi_f(g') = \phi(g'g)$), this is equivalent to saying $k_\theta \cdot \phi_f = \sigma_k(k_\theta)\phi_f$, and $k \cdot \phi_f = \omega(k)\phi_f$, for $k \in K_0(N)$

We want to check whether ϕ_f spans a $(\mathfrak{g},K)$-module: in $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{C})$ we find

$$X_\pm := \frac{1}{2}\begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix}, \tag{7.14}$$

the so-called "raising and lowering operators", which more or less act on ϕ_f as holomorphic and antiholomorphic derivatives. The actions of $\mathfrak{g}$ and $K_0(N)$ are compatible, as $\mathrm{Ad}(k_\theta)X_\pm = e^{\pm 2i\theta}X_\pm$ (Exercise!). Therefore,

$$k_\theta X_\pm \phi_f = (k_\theta X_\pm k_\theta^{-1})k_\theta\phi_f = e^{\pm 2i\theta}\sigma_k(k_\theta)\phi_f = e^{(k\pm 2)i\theta}\phi_f, \tag{7.15}$$

so the matrices $X_\pm$ really raise and lower the weight of ϕ_f.

Exercise 11. *Check the last statement, and also that* $X_-\phi_f = 0$ *if and only if* f *is holomorphic.*

There is another element to account for in $\mathfrak{g}$: as in section 5.2, $h = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, which gives the Casimir element

$$\Delta = -\frac{1}{4}h^2 - \frac{1}{2}(X_+X_- + X_-X_+) \tag{7.16}$$

in the centre of the universal enveloping algebra of $\mathfrak{g}$.

Exercise 12. *Check that*

$$\Delta\phi_f = \frac{-k}{2}(\frac{k}{2} - 1)\phi_f, \tag{7.17}$$

and that Z_∞^+ *acts trivially on* ϕ_f.

It follows that $Z(\mathfrak{g})\phi_f = \mathbb{C}[\Delta]\phi_f$ is one-dimensional.

Definition 7.6. Let $\omega : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ be a Hecke character of conductor dividing N, for some $N \in \mathbb{N}$. An **automorphic form** is a function ϕ satisfying

(i) $\phi(gk_\theta k_0) = \sigma_k(k_\theta)\psi(k_0)\phi(g)$ for all $g \in G(\mathbb{A})$, $k_\theta \in K_\infty^+$ and $k_0 \in K_0(N)$, and

(ii) $\Delta\phi = \lambda\phi$, for some $\lambda \in \mathbb{C}$.

The set of automorphic forms is denoted $A(\psi,\lambda,N,\sigma_k)$.

Definition 7.7. An automorphic form ϕ is **cuspidal** if

$$\int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} \phi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) dx = 0 \tag{7.18}$$

for all $g \in G(\mathbb{A})$.

Remark 7.8. Recall that $f \in S_k(N, \chi)$ being *cuspidal* means that

$$\int_0^1 f(z+t)dt = 0 \quad \text{for all} \quad z \in \mathfrak{h}. \tag{7.19}$$

Defining $n_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for $t \in \mathbb{R}$, note that $\phi_f(n_t g_z) = f|_{n_t}(z) = f(z+t)$. If μ_∞ denotes the Haar measure on $\mathbb{R}_{>0}$ with volume 1, then eq. (7.19) is equivalent to

$$\int_{\mathbb{R}_{>0}} \phi_f(n_t g_z) d\mu(t) = 0. \tag{7.20}$$

More generally, if $g_\infty \in G(\mathbb{R})$, then by the Iwasawa decomposition $G(\mathbb{R}) = B(\mathbb{R})K_\infty^+$, write $g = g_z k_\theta$ for k_θ as before and some $z \in \mathfrak{h}$. Then $\phi_f(n_t g) = \phi_f(n_t g_z)e^{ik\theta}$, so

$$\int_{\mathbb{R}_{>0}} \phi_f(n_t g) d\mu(t) = 0 \tag{7.21}$$

as well. A consequence of this, along with the fact that adelic integrals are defined as products of local integrals, is that cuspidality of f is equivalent to the statement

$$\int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} \phi_f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0 \quad \text{for almost every} \quad g \in G(\mathbb{A}), \tag{7.22}$$

where it is understood that we take the normalised Haar measure on $\mathbb{Q}^\times\backslash\mathbb{A}^\times$. The "almost every" is present in Gelbart, but I don't quite know why it's there.

Let L^2_{cusp} denote the subspace of L^2 consisting of functions which are *cuspidal*, that is, which satisfy eq. (7.18).

Definition 7.9. Let ψ be a Hecke character, fix $\lambda \in \mathbb{C}$ and $N, k \in \mathbb{N}$. A **cuspidal automorphic form** of weight k, level $K_0(N)$, spectral parameter λ and central character ψ, is an element ϕ of $L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}), \psi)$ satisfying

(i) $\phi(gk_\theta k_0) = \sigma_k(k_\theta)\psi(k_0)\phi(g)$ for all $k_0 \in K_0(N)$, $k_\theta \in K_\infty^+$ and $g \in G(\mathbb{A})$;

(ii) $\Delta\phi = \lambda\phi$.

The vector space of such functions is denoted

$$\mathcal{A}_{\mathrm{cusp}}(\psi, \lambda, N, \sigma), \tag{7.23}$$

and is $Z(\mathfrak{g})$-finite and K-finite.

Proposition 7.10. *The assignment $f \mapsto \phi_f$ determines an isomorphism of $G(\mathbb{A})$-modules,*

$$S_2(N,\chi) \to \mathcal{A}_{\mathrm{cusp}}(\psi, -\frac{k}{2}(\frac{k}{2}-1), N, \sigma_k) \subset L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}), \psi), \tag{7.24}$$

The Petersson inner product on $S_k(n,\chi)$ coincides with the natural inner product on L^2.

This is proved in [Gel73], Prop. 3.1 and 3.2.

7.2 Modular forms as automorphic representations

The real power of the adelic theory becomes apparent once we pass from automorphic forms to their associated representations; then we can study automorphic forms using the powerful tools of representation theory. The next three results are the key building blocks in the theory, telling us that automorphic representations match up with classical newforms.

As described in section 5, $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}),\psi)$ decomposes under the action of $G(\mathbb{A})$ as a direct sum of irreducible unitary representations.

Theorem 7.11 (Multiplicity 1)**.** *The multiplicity of each representation is* 1.

Theorem 7.12 (Strong multiplicity 1)**.** *π is determined by π_∞ and a cofinite set of π_p.*

Proof. See [Gel73, Prop. 5.14] or [Cas73]. ∎

If $p \nmid N$, we have natural Hecke operators $T(p)$ on $S_2(N,\chi)$. On the other hand, elements of $\mathcal{A}_{\text{cusp}}(\psi,\lambda,N,\sigma)$ are fixed by K_p for all $p\nmid N$, and so we have an action of the spherical Hecke algebra, hence of $\widetilde{T}(p) = \mathbb{1}_D$ where $D = K_p\begin{pmatrix} p & \\ & 1\end{pmatrix}K_p$.

Exercise 13. *Show that $p^{(k-1)/2}\widetilde{T}(p) = \phi_{T(p)f}$.*

Proposition 7.13. *If f is an eigenform, let $\pi_f := G(\mathbb{A})\phi_f$. Then π_f is irreducible.*

Proof. By strong multiplicity 1, it suffices to show any two irreducible components π' have the same local components away from primes dividing N. Note that π'_p is unramified and irreducible, classified by its Satake parameter $t(\pi_p)$, a semisimple conjugacy class in $\mathrm{GL}_2(\mathbb{C})$. Write

$$t(\pi'_p) = \begin{pmatrix} t_{1,p} & \\ & t_{2,p}\end{pmatrix}. \tag{7.25}$$

Then $\operatorname{tr} t(\pi'_p) == t_{1,p} + t_{2,p} = \lambda_p$, the $T(p)$-eigenvalue of f. Now, any irreducible summand of π_f contains a function with Hecke eigenvalue $p^{(k-1)/2}\lambda_p$. But the minimal polynomial of T_p is $1 - p^{-(k-1)/2}a_pX + \chi(p)X^2 = \det\big(I - t(\pi'_p)X\big)$, so we conclude that the Satake parameter of π'_p is independent of choice of π', hence $\pi'_p = \pi_p$.

Similarly, at infinity, there's a unique π_∞ determined by $\Delta\phi_f = -\frac{k}{2}(\frac{k}{2}-1)$, namely the discrete series representation of weight k, so π_∞ is irreducible. ∎

As a corollary, we get a map sending a newform $f \in S_k(N,\chi)$ to an irreducible constituent of $L^2_0(G(\mathbb{Q})\backslash G(\mathbb{A}),\psi)$. Notice that if f_1 is old, corresponding to a newform f_2, then $\pi_{f_1} \cong \pi_{f_2}$ by strong multiplicity 1, so we get a bijection between newforms and automorphic representations.

Next we define the conductor of π, and show that it coincides with the level of f in the case of $\pi = \pi_f$. Unsurprisingly, we will build it from local data: if $\pi \cong \pi_\infty \otimes \bigotimes' \pi_p$, then the *conductor* of π_p, $c(\pi_p)$, is the minimal p^r such that

$$V_p^{K_0(p^r),\psi} = \left\{v \in V_p : \pi_p(k)v = \psi(k)v \text{ for all } k \in K_0(p^r)\right\}, \tag{7.26}$$

and *global conductor* is $c(\pi) := \prod_p c(\pi_p)$.

Theorem 7.14 (Casselman)**.** *Each $c(\pi_p)$ exists, and* $\dim V_p^{K_0(p^r),\psi} = 1$.

Let $N = c(\pi)$. Then for $K_0(N) \subset K^\infty$, we have $V^{K_0(N),\psi,\sigma_k} = \mathbb{C}\phi$ for some ϕ, depending on π, which corresponds to some ϕ_f. Minimality of N implies f is new at N. As a result, we have a bijection between newforms $f \in S_k(N,\chi)$ and irreducible constituents of $L^2_0(G(\mathbb{Q})\backslash G(\mathbb{A}),\psi)$ with $c(\pi) = N$ and π_∞ is discrete series of weight k. For unramified p, i.e. $p \nmid N$, this is determined by the Satake parameters at p.

Example 7.15. If N is squarefree, χ trivial, then for all $p \mid N$, $\pi_{f,p}$ is (a twist by an unramified quadratic character of?) the Steinberg representation, because this is the only irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ with conductor p.

The following is a consequence of the Weil conjectures:

Theorem 7.16 (Ramanujan–Petersson conjecture)**.** *If $f \in S_k(N, \chi)$, then $|a_p(f)| \leq 2p^{(k-1)/2}$ for all $p \nmid N$.*

This is equivalent to saying $|t_{1,p} + t_{2,p}| \leq 2$, hence $t_{i,p} = p^{s_i}$ for $s_i \in i\mathbb{R}$. Representations satisfying analogues of this property are called *tempered representations*. Thus we can rephrase the conjecture to the following:

Theorem 7.17 (Ramanujan–Petersson conjecture (V2))**.** *For any irreducible constituents π of $L^2_0(G(\mathbb{Q})\backslash G(\mathbb{A}), \psi)$ with π_∞ discrete series of weight k, π_p is tempered.*

This is not known, say, for Maass forms, where π_∞ is principal series.

More generally, we can consider analogues of this for other reductive groups than GL_2, where it is not generally true: we need to specify that π should be *globally generic*, cf. [Gan23].

Part II

Tate's thesis

1 Topological groups

2 Characters of local fields

Speaker: Léo Gratien

In this section, K will denote a local field. As explained in the previous section, its additive group K^+ is a locally compact Hausdorff abelian group, and thus has a Haar measure μ. For any non-zero α in K, we get a new left-invariant measure μ_α on K by

$$\int_K f(x)\mu_\alpha(x) := \int_K f(\alpha x)\mu(x). \tag{2.1}$$

By the uniqueness of μ, $\mu_\alpha = c(\alpha) \cdot \mu$ for some function $c \colon F \setminus \{0\} \to \mathbb{R}$.

Exercise 14. *Check that $c(\alpha)$ defines a norm on K.*

Example 2.1. Let $K = \mathbb{R}$. Then $\mathbb{1}_{[0,\alpha]} = \alpha \mathbb{1}_{[0,1]}$, so $c(\alpha) = |\alpha|$.

Example 2.2. Let $K = \mathbb{C}$. Then $\alpha \mathbb{1}_{[0,1]^2} = \mathbb{1}_{[0,\alpha]^2}$, so $c(\alpha) = |\alpha|^2$.

Example 2.3. Let $K = \mathbb{Q}_p$. Since $\mathbb{1}_{\mathbb{Z}_p} = \sum_{\alpha=0}^{p-1} \mathbb{1}_{\alpha + p\mathbb{Z}_{|p}}$, by translation invariance we see that $p\mu(p\mathbb{Z}_p) = \mu(\mathbb{Z}_p)$, so $c(p) = p^{-1}$.

In a similar way, one shows:

Exercise 15. *Let $K/\mathbb{Q}_p$ be a finite extension with uniformiser ϖ and residue field $F_q \cong \mathcal{O}_K/\varpi$. Show that $c(p) = q^{-1}$.*

This is an analogue of the following characteristic p result:

Exercise 16. *Let $K = \mathbb{F}_q((x))$. Show that $c(x) = q^{-1}$.*

2.1 Characters of $(K, +)$

We will prove the following in a sequence of lemmas:

Theorem 2.4. *Let K be a local field, viewed as an additive group. Then $\hat{K} \cong K$.*

Note first that $\hat{K}$ has a non-zero element χ_0 since $\hat{\hat{K}} \cong K$. Define a map $i : K \to \hat{K}$ by $i(\alpha) = (x \mapsto \chi_0(\alpha x))$.

Lemma 2.5. *The map i is an injective homomorphism, and a homeomorphism onto its image.*

Proof. That i is a homomorphism is clear from the definition. It is injective because χ_0 is nontrivial by assumption. It remains to show that i is continuous and open. Recall that the topology on $\hat{K}$ is generated by open sets of the form $U(C,V)$, where $C \subset K$ is a compact subset, and $V \subset S^1$ is a neighbourhood of 1. Continuity of i means that for any pair C, V there exists $\epsilon > 0$, such that $B(0,\epsilon) \subset i^{-1}(U(C,V))$. But this follows easily from continuity of χ_0.

For openness, we need to show that for any $\epsilon > 0$, there exist C and V such that $i(K) \cap U(C,V) \subset i(B(0,\epsilon))$. I got confused here. ■

Lemma 2.6. *The image $i(K)$ is dense in $\hat{K}$.*

Proof. An easy consequence of Pontryagin duality is that there is an order-reversing bijection between subsets of K and $\hat{K}$, given by $F \mapsto F^\perp := \{\chi \in \hat{K} : \chi(F) = \{1\}\}$. As before, $\{x \in K : i(\alpha)(x) = 1\ \forall \alpha \in K\} = \{1\}$, so $\overline{i(K)}$ corresponds to $\{1\}$. ■

By the usual topological argument, theorem 2.4 follows from the final lemma:

Lemma 2.7. *The image $i(K)$ is closed in $\hat{K}$.*

Proof. Pick a sequence $(x_n = \chi_0(\alpha_n \cdot)) \in i(K)^{\mathbb{N}}$ such that $i(x_n) \to \psi$ for some $\psi \in \hat{K}$. Note that (α_n) is Cauchy by an argument similar to that of openness of i, so (α_n) converges to some element $\alpha \in F$. We claim that $\psi = \chi_0(\alpha \cdot)$. But this follows from continuity of χ_0. ■

Just like for vector spaces, the different choices of χ_0 give different isomorphisms between K and its dual. However, in our examples from above, there are certain more or less standard choices:

Example 2.8. Let $K = \mathbb{R}$. Then we can take $\chi_0(x) = e^{-2\pi x}$.

Example 2.9. Let $K = \mathbb{C}$. Then we can take $\chi_0(z) = e^{-2\pi i \Re(z)}$.

Example 2.10. Let $K = \mathbb{Q}_p$. If $a = \sum_{i \gg -\infty} a_i x^i$, write $q(a) := a = \sum_{i<1} a_i x^i$ for the "fractional part" of a. Then $\chi_0(a) = e^{2\pi i q(a)}$ defines an additive character of K.

Example 2.11. Let $K/\mathbb{Q}_p$ be a finite extension. Then we can take $\chi_0(a) = e^{2\pi i q(\operatorname{Tr} a)}$ where q is as in the preceding example.

Example 2.12. Let $K = F_q((x))$. For $a = \sum_{i \gg -\infty} a_i x^i$, set $\chi_0(a) = \exp\left(2\pi i \frac{\operatorname{Tr} a_{-1}}{p}\right)$, where Tr denotes the trace map from $\mathbb{F}_q$ to $\mathbb{F}_p$.

2.2 Quasicharacters of $(K^\times, \times)$

It will be clear in later sections that looking only at *unitary* characters of $K^\times$, meaning characters valued in S^1 instead of $\mathbb{C}$, is too restrictive. Therefore we make the following definition:

Definition 2.13. Let K be a local field. A **quasicharacter of** K is a continuous group homomorphism $\chi\colon K^\times \to \mathbb{C}^\times$. A quasicharacter χ is **unitary** if $|\chi(x)| = 1$ for all $x \in F^\times$.

Of course, a unitary quasicharacter is simply a character in the above sense.

Theorem 2.14. *Any quasicharacter can be decomposed $\chi = \chi_0 \cdot |-|^s$ for some $s \in \mathbb{C}$, where χ_0 is unitary.*

This is not particularly hard: first one shows that the only character χ which is unramified (i.e. trivial on $\mathcal{O}_F^\times$) is $x \mapsto |x|^s$. Therefore, $\chi \cdot |-|^s$ will be unitary for some s. Furthermore, $s \in \mathbb{C}$ is unique if K is archimedean, and unique up to multiples of $2\pi i / \log q$ if the residue field of K has order q.[1]

[1]What if the residue field is infinite?

Example 2.15. Let $K = \mathbb{R}$. Then $K^\times = \{\pm 1\} \times \mathbb{R}_{>0}$.

Example 2.16. Let $K = \mathbb{C}$. Then $K^\times = S^1 \times \mathbb{R}_{>0}$.

Example 2.17. Let $K = \mathbb{Q}_p$. Then $K^\times = p^{\mathbb{Z}} \cdot \mathbb{Z}_p^\times$.

Example 2.18. Let $K/\mathbb{Q}_p$ be a finite extension. Then $K^\times = \varpi^{\mathbb{Z}} \times \mathcal{O}_F^\times$.

Exercise 17. *Decompose $K^\times$ when $K = \mathbb{F}_q((x))$.*

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